Estimation of the Dependence Parameter in Bivariate Archimedean Copula Models Under Misspecification

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Abstract

Copulas have become increasingly popular in multivariate statistics and financial applications. This paper studies the effect of misspecification among the three Archimedean copula families Frank, Gumbel and Clayton on the dependence parameter estimation for two dimensions. In addition to the maximum likelihood estimator and the inverted Kendall's tau estimator, a p-value weighted average of the two is proposed and studied. To assess the performance of the proposed estimator, a comprehensive simulation study was conducted. As opposed to ML, the suggested estimator is shown to yield unbiased results even under copula misspecification for certain combinations of true copula, misspecified copula and dependence level. In the given application of estimating the Value-at-Risk of two bivariate portfolios using the three Archimedean copulas in combination with each of the three estimators, the proposed estimator also outperformed the ML estimator on the whole.

Keywords: *Misspecification, Archimedean copula, Multivariate dependence, Copula estimation, Simulation study, Value-at-Risk*

Copulas erfreuen sich immer größer werdender Beliebtheit in der multivariaten Statistik und im Anwendungsbereich der Finanzwissenschaft. Diese wissenschaftliche Arbeit untersucht die Auswirkungen von Misspezifikation der Copula auf die Schätzung des Abhängigkeitsaparameters in Zusammenhang mit den drei archimedischen Copula-Familien Frank, Gumbel und Clayton. Neben dem Maximum-Likelihood-Schätzer und dem Schätzer basierend auf Kendalls Tau, wird ein neuer Schätzer in Form eines mit dem p-Wert gewichteten Durchschnitts der beiden vorher genannten Schätzer vorgeschlagen und untersucht. Um die Performance des Schätzers zu evaluieren, wurde eine umfassende Simulationsstudie durchgeführt. Es wird gezeigt, dass der vorgeschlagene Schätzer, im Gegensatz zum Maximum-Likelihood-Schätzer, trotz misspezifizierter Copula unverzerrte Ergebnisse für bestimmte Kombinationen von wahrer Copula, misspezifizierter Copula und Abhängigkeitsniveau liefert. In der gegebenen Anwendung wird der Value-at-Risk zweier bivariater Portfolios mit Hilfe der drei archimedischen Copulas in Kombination mit jedem der drei Schätzer geschätzt. Die Backtesting-Ergebnisse zeigen, dass der vorgeschlagene Schätzer insgesamt bessere Ergebnisse liefert als der Maximum-Likelihood-Schätzer.

Schlagwörter: *Misspezifikation, Archimedische Copula, Multivariate Abhängigkeit, Copula-Schätzung, Simulationsstudie, Value-at-Risk*

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List of Abbreviations

C	Clayton
CDF	Cumulative Distribution Function
Clay	Clayton
CML	Canonical Maximum Likelihood
DAX	Deutscher Aktienindex
DJ(IA)	Dow Jones Industrial Average
EC	Estimation Copula
ECDF	Empirical Distribution Function
F	Frank
FML	Full Maximum Likelihood
Fra	Frank
G	Gumbel
GMM	Generalised Method of Moments
GoF	Goodness-of-Fit
Gum	Gumbel
i.i.d.	independent and identically distributed
IFM	Inference for Margins
KL	Kullback-Leibler
KS	Kolmogorov-Smirnov
MM	Method of Moments
ML(E)	Maximum Likelihood (Estimator)
MSE	Mean Squared Error
PRB	Percentage Relative Bias
P&L	Profit and Loss
PML(E)	Pseudo-Maximum Likelihood (Estimator)
QML(E)	Quasi-Maximum Likelihood (Estimator)
SGED	Skewed Generalised Error Distribution
TK	Thyssen-Krupp
TC	True Copula
VaR	Value at Risk
VW	Volkswagen

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1 Introduction

In statistical modelling, the validity of the results hinges on the question of adequate model specification. In practice, all models are "misspecified", but it is assumed that the specified model is close to the "true" model and approximates it well enough, such that correct inferences can be made. In fact, a quote of the statistician George E. P. Box gets to the heart of statistical modelling: "All models are wrong, though some are useful." Usefulness again depends on the purpose. As we will see, a model misspecified in the classical, statistical sense can indeed be useful.

Modelling multivariate dependence is of major importance in many research fields such as actuarial sciences, finance and hydrology. For this purpose, copulas have become increasingly popular (see e.g. Cherubini et al., 2004; Salvadori et al., 2007; Frees and Valdez, 1998). Nelsen (2006, p. 1) defines copulas as link functions "that join or couple multivariate distribution functions to their one-dimensional marginal distribution functions". Copulas owe their popularity to the flexibility and possibilities they offer for modelling multivariate dependence. Before the discovery of copulas, multivariate dependence was simply modelled by imposing the multivariate normal or the multivariate t-distribution. These, however, imply symmetry, linear dependence, not too heavy tails and require the univariate margins to also be normally or t-distributed. Empirical evidence, however, contradicts these assumptions in many applications.

The purpose of this thesis is to study the effect of copula misspecification among the three Archimedean copula families Frank, Gumbel and Clayton on the estimation of the copula dependence parameter in two dimensions. In particular, a new estimator is proposed that is meant to do the following: having the maximum likelihood estimator and the method of moments estimator based on Kendall's τ at hand, it selects the estimator that is more "useful" for the purpose of estimating the dependence parameter θ. The idea and intention will be further clarified by means of an example after the literature review.

Being rooted in Sklar's Theorem Sklar (1959) as early as 1959, copulas have only appeared more frequently in the literature starting from 1999, as the analysis of Genest et al. (2009a) shows. Sklar's Theorem decomposes the joint distribution of continuous random variables into their marginal distributions and their dependence structure modelled by the copula function:

$$F(y_1, \ldots, y_d) = C\{F_1(y_1), \ldots, F_d(y_d)\}$$

This decomposition facilitates model specification in the sense that univariate marginals and dependence structure may be modelled separately. This not only allows using varying univariate marginal distributions, but also permits the modelling of heavy tails and more sophisticated dependence structures.

Copula functions can be differentiated into numerous classes and families. The books of Joe (1997, 2014) and Nelsen (2006) provide an elaborate review of these. An extensively studied class of copulas are Archimedean copulas. Well-known Archimedean copula families are the Frank (Frank, 1979), the Gumbel (Gumbel, 1960) and the Clayton (Clayton, 1978) copula family. Their copula functions are known up to one parameter that needs to be estimated from the data. As such, the Frank, the Gumbel and the Clayton copula rank among the class of one-parameter copulas. The Frank copula is studied in Genest (1987), the Gumbel copula is examined in Hougaard (1986) and for Clayton Cook and Johnson (1981), Oakes (1982, 1986), Cox and Oakes (1984), and Cook and Johnson (1986) are important references.

The estimation of copulas by means of maximum likelihood (ML) can be computationally demanding, especially in high dimensions. Thus, the two-stage Inference for Margins (IFM) procedure has been advanced by Joe and Xu (1996). Drawing on the empirical distribution function to model the margins, Genest et al. (1995) have introduced a semiparametric estimation procedure, the so called canonical maximum likelihood (CML). Additional semiparametric estimators have been suggested by Tsukahara (2005), namely a rank approximate Z-estimator and a minimum distance estimator. Mostly used in the bivariate one-parameter case, a method-of-moments estimation procedure exploiting the functional relationship between the copula dependence parameter and a measure of rank-correlation, for example Kendall's τ, was studied in Oakes (1982), Genest (1987), and in Genest and Rivest (1993).

Any misspecification of the parametric structure of the copula function may impact estimation results and inference. Numerous Goodness-of-Fit tests have therefore been suggested to check the adequacy of the assumed copula. Tests based on the empirical copula were promoted by Genest and Rémillard (2008). Alternatively, tests based on Kendall's process (see e.g. Genest and Rivest, 1993; Wang and Wells, 2000) or on the Rosenblatt transform (see e.g. Rosenblatt, 1952; Breymann et al., 2003; Dobrić and Schmid, 2007) have been proposed. A review of the afore-mentioned tests along with a power study can be found in Genest et al. (2009b).

Various aspects of misspecification in the context of copulas have been examined. To name some examples, Fermanian and Scaillet (2005) find that parametric misspecification of the marginals may entail severely biased estimates of the dependence parameter and remark that the efficiency loss of the semiparaemtric approach compared to ML is small in large samples. Kim et al. (2007) affirm these findings by showing that ML and IFM are non-robust against misspecified margins and that CML dominates IFM and MLE if margins are unknown. However, Prokhorov and Schmidt (2006) show that radially symmetric copulas are robust against misspecification in problems about sample means if the true joint densities also radially symmetric.

Copulas are widely used to model dependence in the fields of asset pricing and risk management (see e.g. Cherubini et al., 2004, 2012). Financial modelling is one of the examples where the implications of the multivariate normal distribution are inappropriate. In fact, the normality assumption was rejected in many empirical studies, for example by Fama and French (1993), Richardson and Smith (1993), Longin and Solnik (2001), and by Mashal and Zeevi (2002) among others. Ample empirical evidence has documented that financial returns follow skewed and heavy-tailed distributions (see e.g. McNeil et al., 2005). This as well as the need to model the dependence between extreme values of various assets gave rise to the extensive use of copulas in finance. In particular, copulas may be employed for the estimation of the Value at Risk (VaR) of a multivariate portfolio (see e.g. Cherubini et al., 2012; Giacomini et al., 2009). Misspecification has also been a subject of study in the VaR context. Fantazzini (2009) examines the effect of misspecified marginals and copulas on the estimation of the Value at Risk.

	Min.	Mean	Max.	Bias
$\hat{\theta}_{ML}$	1.574	1.744	1.917	-0.256
		0.068		
$\hat{\theta}_{\tau}$	1.775	2.019	2.327	0.019
		(0.097)		
$\hat{\theta}_{ML} - \hat{\theta}_{\tau}$	-0.425	-0.276	-0.174	-
		0.042		

Table 1: Summary statistics of the estimates for θ obtained by assuming a Gumbel copula and applying maximum likelihood as well as the inversion of Kendall's τ to a random sample from a Clayton copula with dependence parameter $\theta = 2$. The difference between the estimates of the two estimators is summarised below. Second lines contain the corresponding standard deviations.

COPmisspecClayGum

Motivating Example

To see the motivation behind the proposed estimator, consider the following example.

Step 1: Simulate 150 random samples of size 500 from a bivariate Clayton copula with dependence parameter $\theta = 2$, corresponding to a Kendall's $\tau = 0.5$.

Step 2: Estimate θ by maximum likelihood ($\hat{\theta}_{ML}$) and by the inversion of Kendall's τ ($\hat{\theta}_{\tau}$).

The results are summarised in Table 1. First of all, note that the two estimates obtained by maximum likelihood (ML) and the inversion of Kendall's τ differ on average by 0.276 in absolute terms. While the maximum likelihood estimator (MLE) constantly underestimates the true θ, the inversion of Kendall's τ, on average, renders a correct result despite the use of a misspecified copula.

In case the true parameter is unknown, it is not clear which estimate is closer to the value of the true parameter. The rather big difference between the two estimates may, however, be taken as hint to misspecification. The two estimates tend to be closer if the true copula is used for estimation, which is also evident from the results of Section 4. The adequacy of the assumed parametric copula can be tested through employing a Goodness-of-Fit (GoF) test. In case a low p-value points to a misspecified copula, Kendall's τ seems to be the better choice. To wrap this up in one, we can construct a p-value weighted average of the two estimators the following way:

$$\hat{\theta}_{new} = p \cdot \hat{\theta}_{ML} + (1 - p) \cdot \hat{\theta}_{\tau} \,, \tag{1}$$

where p is the p-value obtained testing the model estimated by ML. In case of a misspecified copula, the p-value of the GoF test will be small and only little weight will be given to the ML estimate. In contrast, a lot of weight will be placed on the inverted Kendall's τ estimate, which seems to be more robust against misspecification compared to the MLE. The contrary is true in case of a well-specified copula. The p-value will be high and the ML estimate will dominate, which is known to be consistent and efficient if the assumed parametric model corresponds to the true one.

The purpose of this paper is to investigate the performance of this estimator by means of a simulation study involving the Frank, the Gumbel and the Clayton copula as well as different dependence levels. Furthermore, the estimator is applied to real data and used to estimate the Value at Risk of two bivariate portfolios.

The proposed estimator is shown to yield unbiased results even under copula misspecification, if the sample originates either from a Gumbel or a Clayton copula with dependence level $\tau = 0.5$ and the respective other copula is erroneously employed for estimation. Furthermore, it is shown to return the less biased Kendall's τ estimate if either Clayton or Gumbel are used as misspecified copula in estimation, independent of which of the three copulas is the true one. Using simply Gumbel or Clayton to estimate θ from data stemming from convex sums of Gumbel and Clayton, the proposed estimator returns, as opposed to ML, correct results. Also in the empirical application, the proposed estimator overall performs better than ML.

The remainder of this thesis is structured as follows. Section 2 revises some relevant copula theory and provides the theoretical background for the proposed estimator $\hat{\theta}_{new}$. Section 3 then discusses the estimation of the Value-at-Risk using copulas. The results of the simulation study are presented in Section 4 and in Section 5 the estimator is applied to real world data. Finally, Section 6 summarises the results and draws conclusions.

2 Review of Copulas and Theoretical Background of the Proposed Estimator

For a d-variate, continuous random variable $\mathbf{Y}$ with distribution function $F \in \mathcal{F}(F_1, \ldots, F_d)$ and F_j being the jth univariate, continuous margin, the copula associated with F is a distribution function $C : [0,1]^d \to [0,1]$ with $U(0,1)$ margins that satisfies

$$F(\boldsymbol{y}) = C\left(F_1(y_1), \ldots, F_d(y_d)\right), \quad \boldsymbol{y} \in \mathbb{R}^d \tag{2}$$

(Sklar, 1959; Joe, 2014). If the univariate margins are absolutely continuous with respective densities $f_j = F_j'$ and if C has mixed derivatives of order d, the joint density function of the multivariate distribution F is given by

$$f(\boldsymbol{y}) = c\left(F_1(y_1), \ldots, F_d(y_d)\right) \times \prod_{j=1}^{d} f_j(y_j), \quad \boldsymbol{y} \in \mathbb{R}^d \tag{3}$$

with $c(\boldsymbol{u}) = c(u_1, \ldots, u_d) = \partial^d C(\boldsymbol{u})/\partial u_1, \ldots, u_d,\ \boldsymbol{u} \in [0,1]^d$ denoting the copula density of $C(\cdot)$ (Joe, 2014, pp. 7).

To arrive at a copula model for a multivariate random variable $\boldsymbol{Y}$, C is assumed to belong to a parametric family

$$C \in \{C_\theta,\ \theta \in \Theta\},$$

where $\Theta \subset \mathbb{R}^p$ is a p-dimensional parameter space (see e.g. Zhang et al., 2013).

The flexibility of copulas is not restricted to modelling the margins with different distributions. Also different copula classes and families can be mixed. A convex combination of copulas is again a copula. Formally, if C_1 and C_2 are d-variate copulas, then

$$C = \alpha C_1 + (1-\alpha) C_2, \quad \alpha \in [0,1] \tag{4}$$

is also a copula (see e.g. Mikusiński et al., 1991). This is useful for combining features of one copula family or class with characteristics of another copula family or class.

2.1 Archimedean Copulas

A popular and extensively studied class of copulas are the Archimedean copulas. In contrast to elliptical copulas, like the Gaussian or the Student's t-copula, Archimedean copulas are not constructed according to Sklar's Theorem in Equ. (2). Instead, they are based on the Laplace transforms ϕ of univariate distribution functions. A d-dimensional, exchangeable Archimedean copula is defined as

$$C(u_1, \dots, u_d) = \phi\{\phi^{-1}(u_1) + \cdots + \phi^{-1}(u_d)\}, \quad u_1, \dots, u_d \in [0,1], \tag{5}$$

where $\phi \in \mathcal{L}$, $\phi : [0, \infty] \to [0, 1]$ is called the generator of the copula and depends on θ (see e.g. Härdle and Okhrin, 2010). $\mathcal{L}$ denotes the class of Laplace transforms consisting of strictly decrasing, differentiable functions. For a non-negative random variable Y the Laplace transform is defined as

$$\phi(s) = \phi_Y(s) := \mathrm{E}[e^{-sY}] = \int_{[0,\infty)} e^{-sy} dF_Y(y), \quad s \geq 0.$$

For $\phi \in \mathcal{L}$, $\phi(0) = 1$, $\phi(\infty) = 0$, and $(-1)^j \phi^{(j)}(s) \geq 0$ for $j = 1, 2, \dots, \infty$ and $s > 0$, where $\phi^{(j)}(s)$ denotes the *jth* derivative. That is to say, ϕ has continuous derivatives of all orders that alternate in sign (see e.g. Joe, 2014, p. 33).

Note that some authors refer to $\psi := \phi^{-1}$ as the generator of the Archimedean copula and define Equ. (5) as $C(u_1, \dots, u_d) = \psi^{-1}\{\psi(u_1) + \cdots + \psi(u_d)\}$. Analytically, this is of course identical since $\psi^{-1} = (\phi^{-1})^{-1} = \phi$.

Three popular one-parameter Archimedean families are the Frank, the Gumbel, and the Clayton family. They are discussed in the following subsections. The overview is based on Joe (2014) and Härdle and Okhrin (2010). As this thesis focuses on two dimensions, copula functions are given for the bivariate case.

2.1.1 Frank Copula (Frank, 1979)

The Frank copula is the only Archimedean copula with radial symmetry. Radial symmetry means that the copula equals its survival copula $\bar{F}(\boldsymbol{y}) = \widehat{C}\left((\bar{F}_1(y_1), \dots, \bar{F}_d(y_d)\right)$ (see e.g. Joe, 2014, p.8). Generator and copula function are:

$$\phi(s; \theta) = -\theta^{-1} \log\left\{1 - (1 - e^{-\theta})e^{-s}\right\}, \quad \theta \in [0, \infty), \quad s \in [0, \infty), \tag{6}$$

$$C(u, v; \theta) = -\theta^{-1} \log \left\{ 1 - \frac{(1 - e^{-\theta u})(1 - e^{-\theta v})}{1 - e^{-\theta}} \right\}, \quad 0 \leq u, v \leq 1. \tag{7}$$

Setting $\theta = 0$ gives the independence copula. For $\theta \to \infty$, the dependence becomes maximal.

2.1.2 Gumbel Copula (Gumbel, 1960)

The Gumbel copula is often used to model dependence in financial applications. Its generator and copula function are:

$$\phi(s; \theta) = \exp\left\{ -s^{1/\theta} \right\}, \quad \theta \in [1, \infty), \quad s \in [0, \infty), \tag{8}$$

$$C(u, v; \theta) = \exp\left[-\left\{ (-\log\, u)^{\theta} + (-\log\, v)^{\theta} \right\}^{1/\theta} \right], \quad 0 \leq u, v \leq 1. \tag{9}$$

The Gumbel copula has dependence in its upper tail characterised by the upper tail coefficient $\lambda_u = 2 - 2^{1/\theta}$. No dependence and more variability as well as more mass characterise its lower tail. This leads to asymmetric contour plots. Upper tail dependence is achieved for $\theta > 1$. The limiting cases for $\theta \to 1$ and $\theta \to \infty$ are the independence copula and perfect positive dependence, the so-called Fréchet-Hoeffding upper bound, respectively.

The Gumbel copula is the only Archimedean copula that can be used to construct an extreme value distribution. A bivariate distribution with univariate extreme value marginal distributions and Gumbel dependence structure is the only extreme value distribution whose copula function is Archimedean and, under common regularity conditions, all distributions with Archimedean dependence functions belong to its domain of attraction. These two results were shown by Genest and Rivest (1989).

2.1.3 Clayton Copula (Clayton, 1978)

The Clayton copula is, in contrast to the Gumbel family, characterised by lower tail depencence determined according to $\lambda_l = 2^{-1/\theta}$, more mass in the lower tail and less mass in the upper tail. Its generator and copula function are:

$$\phi(s; \theta) = (1 + \theta s)^{-\frac{1}{\theta}}, \quad \theta \in [-1, 0) \cup (0, \infty), \quad s \in [0, \infty) \tag{10}$$

$$C(u, v; \theta) = (u^{-\theta} + v^{-\theta} - 1)^{-\frac{1}{\theta}}, \quad 0 \leq u, v \leq 1. \tag{11}$$

For $\theta \to \infty$, the distribution approaches the Fréchet-Hoeffding upper bound, for $\theta \to 0$ the independence copula is obtained, and for $\theta \to -1$ perfect negative dependence results, that is to say the distribution tends to the Fréchet-Hoeffding lower bound.

Note that the lower bound of -1 for θ is specific to the bivariate case. In general, the lower bound for θ for a d-variate distribution is $-1/(d-1)$, $\theta \neq 0$.

To conclude the subsection about Archimedean copulas, it shall be remarked that a convex combination of two Archimedean copulas according to Equ. (4) is again an Archimedean copula (Nelsen, 2006; Pfaff, 2013). This result will be utilised in Subsection 4.2 to construct a mixture of the Clayton and the Gumbel copula using different values of α. When using a pure Clayton or Gumbel copula, either lower or upper tail dependence can be modelled. The advantage of using a convex combination of the copulas is that both lower and upper tail dependence can be modelled simultaneously. Lower and upper tail dependence coefficients are then given by $\lambda_l = \alpha 2^{-1/\theta_C}$ and $\lambda_u = (1-\alpha)(2 - 2^{1/\theta_G})$, respectively, where θ_G and θ_C denote the respective dependence parameter of the used Gumbel and Clayton copula (Pfaff, 2013, p. 149).

2.2 Estimation and Calibration of Copulas

Once the copula model has been selected, three maximum likelihood based methods can be employed to calibrate the model. The first one is the full maximum likelihood (FML), which estimates the copula parameter θ and the parameters of the univariate margins δ simultaneously. The second one estimates the parameters for the margins and the copula parameter in two stages. This method is referred to as Inference for Margins (IFM). The third method uses the empirical distribution function to model the margins and then estimates the copula parameter. For Archimedean copulas, a method-of-moments estimation based on Kendall's τ was proposed by Genest and MacKay (1986a) and is studied in Genest and Rivest (1993). The purpose of this section is to revise these estimation procedures.

2.2.1 Full Maximum Likelihood

Given the parametric model for the marginals and for the copula, the parameters can be estimated in one step under $H_0 : F_1 \in \mathcal{F}_1, \ldots, F_d \in \mathcal{F}_d$, $C_0 \in \{C_\theta,\, \theta \in \Theta\}$ through

$$\hat{\alpha} = \arg\max_{\alpha} \ell(\alpha),$$

where $\hat{\alpha} = (\hat{\delta}_1, \ldots, \hat{\delta}_d, \hat{\theta})^\top \in \mathbb{R}^{d+1}$ and $\hat{\delta}_j$, $j = 1, \ldots, d$ denote estimates for the parameters of the marginal distributions. For a sample size of n observations, the log-likelihood function has the form

$$\ell(\alpha;\, y_1, \ldots, y_n) = \sum_{i=1}^{n} \log c\{F_1(y_{1i};\, \delta_1), \ldots, F_d(y_{di};\, \delta_d);\, \theta\} + \sum_{i=1}^{n}\sum_{j=1}^{d} \log f_j(y_{ji};\, \delta_j). \tag{12}$$

Under H_0 and the usual regularity conditions of ML (see e.g. Serfling, 1980) being fulfilled for the multivariate model as well as for the margins, FML is consistent, asymptotically efficient, and $\sqrt{n}(\hat{\alpha} - \alpha)$ is asymptotically normal with zero mean. However, it might be computationally demanding, especially in higher dimensions.

2.2.2 Inference for Margins

To alleviate the computational burden of FML, Joe and Xu (1996) proposed to maximise the log-likelihood in Equ. (12) in two steps. First, the $j = 1, \ldots, d$ log-likelihood functions of the margins in the latter part are optimised to obtain estimates for the parameters of the margins δ_j, $j = 1, \ldots, d$. The estimates are then inserted into the first part of Equ. (12). In the second step, the resulting pseudo-log-likelihood function is maximised over θ:

$$\hat{\theta}_{IFM} = \arg\max_{\theta}\, \ell(\theta; \hat{\delta}_1, \ldots, \hat{\delta}_d) = \arg\max_{\theta} \sum_{i=1}^{n} \log c\{F_1(y_{1i};\, \hat{\delta}_1), \ldots, F_d(y_{di};\, \hat{\delta}_d); \theta\}. \tag{13}$$

Under some additional regularity conditions, $\sqrt{n}(\hat{\theta}_{IFM} - \theta)$ converges to a normal distribution with zero mean (Joe, 1997, 2005). IFM eases computational effort, however, comes at the cost of efficiency loss. Joe (1997) though argues that IFM is nonetheless highly efficient compared with FML. An additional analysis and some exceptions are provided in Joe (2005).

2.2.3 Canonical Maximum Likelihood

The Canonical Maximum Likelihood (CML) draws on the empirical cumulative distribution function (ecdf) $\hat{F}$ to estimate the margins instead of assuming a parametric model. This implies that the copula parameter may be estimated without specifying the marginals. For each margin, the empirical distribution function is defined as

$$\hat{F}_j(y) = \frac{1}{n+1} \sum_{i=1}^{n} \mathbf{I}(y_{ji} \leq y).$$

The empirical cdfs are used to transform the sample data $\{y_{1i}, y_{2i}, \ldots, y_{di}\}_{i=1}^{n}$ into a sample of pseudo-observations in the d-unit cube $\{u_{1i}, u_{2i}, \ldots, u_{di}\}_{i=1}^{n}$. The uniform variates are then used to estimate the copula parameter using ML

$$\hat{\theta}_{CML} = \arg\max_{\theta} \ell(\theta) = \arg\max_{\theta} \sum_{i=1}^{n} \log c\{\hat{F}_1(y_{1i}), \dots, \hat{F}_d(y_{di}); \theta\} \tag{14}$$

Under certain regularity conditions, $\sqrt{n}(\hat{\theta}_{CML} - \theta)$ is asymptotically normal with mean zero (Genest et al., 1995). However, the estimator is not asymptotically semi-parametrically efficient in general. Some exceptions are demonstrated in Genest and Werker (2002).

2.2.4 Inversion of Kendall's τ

To calibrate a bivariate Archimedean copula, a procedure based on sample dependence measures was proposed by Genest and MacKay (1986a), (see also Oakes, 1982; Genest and MacKay, 1986b; Genest, 1987; Genest and Rivest, 1993). The method is very simple, however, in its simplicity limited to the bivariate case as it relies on the dependence coefficients. In this paper, Kendall's τ is chosen to be that dependence measure for two reasons: it proved to perform significantly better than Spearman's ρ (see e.g. Kojadinovic and Yan, 2010a) and the expression of Kendall's τ is explicit for all three considered copula families.

For two continuous random variables (X, Y), a pair of observations is concordant if $x_i < x_j \wedge y_i < y_j$ or if $x_i > x_j \wedge y_i > y_j$. Conversely, it is said to be discordant if $x_i < x_j \wedge y_i > y_j$ or if $x_i > x_j \wedge y_i < y_j$ (Nelsen, 2006, pp.158). In general, the sample version $\hat{\tau}$ of Kendall's τ is defined in terms of concordant pairs c and discordant pairs d:

$$\hat{\tau} = \frac{c - d}{c + d}$$

The population version τ of Kendall's τ for a vector (X, Y) of continuous random variables with a joint distribution function F can be defined as

$$\tau = \tau_{X,Y} = \mathrm{P}\{(X_1 - X_2)(Y_1 - Y_2) > 0\} - \mathrm{P}\{(X_1 - X_2)(Y_1 - Y_2) < 0\}, \tag{15}$$

where (X_1, Y_1) and (X_2, Y_2) are i.i.d. random vectors. As shown in Nelsen (2006, pp.158), Equ. (15) can be be written as

$$\tau_{X,Y} = \tau_C(\theta) = 4 \iint_{[0,1]\times[0,1]} C(u, v) dC(u.v) - 1 = 4\,\mathrm{E}\left[C(U, V)\right] - 1 \tag{16}$$

for two continuous random variables X and Y whose copula is C. Equ. (16) can also be

Archimedean family	Kendall's τ
Frank	$1 + 4\,\theta^{-1}\{D_1{}^{*}(\theta) - 1\}$
Gumbel	$(\theta - 1)/\theta$
Clayton	$\theta/(\theta + 2)$

* D_1 is the Debeye function of order 1, $D_k(x) = kx^{-k}\int\limits_0^x t^k(e^t-1)^{-1}dt$ for $k = 1, 2$

Table 2: Kendall's τ as a function of the copula dependence parameter θ.

interpreted as the expected value of the joint distribution function $C(U, V)$ of the transformed random variables $U = F_X(X)$ and $V = F_Y(Y)$ that are uniform on $[0, 1]$ and whose copula is C. $\tau_C(\theta) = \tau_{X,Y}$ can be consistently estimated by

$$\hat{\tau}_{X,Y} = \frac{4}{n(n-1)}\sum_{i\neq j}\mathbf{I}(X_i \leq X_j)\,\mathbf{I}(Y_i \leq Y_j) - 1. \tag{17}$$

For an Archimedean copula, there is a way to circumvent evaluating the double integral in Equ. (16) using the inverse of the generator function (Genest and MacKay, 1986a,b)

$$\tau_C(\theta) = 1 + 4\int\limits_0^1 \frac{\psi(s)}{\psi'(s)}ds, \quad \textit{where } \psi := \phi^{-1}. \tag{18}$$

If the population version of τ can be expressed as a one-to-one function of the copula dependence parameter θ, a consistent estimator of θ is given by

$$\hat{\theta}_\tau = \tau^{-1}(\hat{\tau}).$$

The $\tau(\theta)$ functions used for inference are given in Table 2 for each of the three considered copulas. $\sqrt{n}(\hat{\theta}_\tau - \theta)$ is asymptotically normal with zero mean (see e.g. Kojadinovic and Yan, 2010a). Further note that, opposed to the previously discussed estimation methods, marginal distributions need not be modelled to obtain an estimate of θ.

2.3 Goodness-of-Fit Test

To check the adequacy of the assumed parametric copula family, that is to say to test $H_0 : C \in \{C_\theta,\ \theta \in \Theta\}$ against $H_1 : C \notin \{C_\theta,\ \theta \in \Theta\}$, numerous Goodness-of-Fit tests (GoF tests) have been proposed for copulas. As already mentioned in the introduction, an overview as well as a power study can be found in Genest et al. (2009b). This section will introduce the GoF test used to obtain the p-value p in Equ. (1).

A natural way to test H_0 is to evaluate the distance between a nonparametric estimate $\hat{C}_n$ and the parametric estimate $C_{\hat{\theta}_n}$ of C. Thus, the test is based on the empirical process

$$\mathbb{C}_n(\boldsymbol{u}) = \sqrt{n}\left(\hat{C}_n(\boldsymbol{u}) - C_{\hat{\theta}_n}(\boldsymbol{u})\right), \quad \boldsymbol{u} = (u_1, \ldots, u_d) \in [0,1]^d,$$

where $\hat{C}_n(\boldsymbol{u})$ denotes the so-called empirical copula. It is defined as

$$\hat{C}_n(\boldsymbol{u}) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{I}(\hat{U}_{i1} \leq u_1, \ldots, \hat{U}_{id} \leq u_d) \tag{19}$$

and is a consistent estimator of C under minimal regularity conditions (see e.g. Gänssler and Stute, 1987; Fermanian et al., 2004; Tsukahara, 2005; Segers, 2012). $\hat{\boldsymbol{U}}_1 = (\hat{U}_{11}, \ldots, \hat{U}_{1d}), \ldots, \hat{\boldsymbol{U}}_n = (\hat{U}_{n1}, \ldots, \hat{U}_{nd})$ denote pseudo-observations deduced from ranks

$$\hat{U}_{ij} = \frac{1}{(n+1)} R_{ij} = \frac{n}{(n+1)} \hat{F}_j(Y_{ij}), \quad i \in \{1, \ldots, n\}, \quad j \in \{1, \ldots, d\}.$$

The name pseudo-observations refers to the fact that the actual observations $\boldsymbol{U}_i = \{F_1(Y_{i1}), \ldots, F_d(Y_{id})\}_{i=1}^n$ are not observable since the marginal distribution functions $F_1 \ldots, F_d$ are usually unknown and have to be estimated by the empirical cdf. $n/(n+1)$ is an asymptotically negligible scaling factor preventing the copula density c_θ from blowing up near the boundary of $[0,1]^d$. The pseudo-observations can be interpreted as a sample from the underlying copula (see e.g. Genest et al., 2013). However, for inference, it is important to take into account that they are only approximately uniform on $[0,1]$ and not mutually independent.

As a test statistic, a rank-based version of the Cramér-von Mises statistic is constructed

$$S_n = \int_{[0,1]^d} \mathbb{C}_n(\boldsymbol{u})^2 d\hat{C}_n(\boldsymbol{u}). \tag{20}$$

Large values of S_n imply a great disparity between the parametric and the empirical copula and lead to a rejection of H_0. The limiting distribution of S_n depends on the copula family under the null hypothesis and on the unknown, true parameter θ. As such, p-values always need to be approximated using a parametric bootstrap procedure (see e.g. Genest and Rémillard, 2008; Genest et al., 2009b), which was proven to be valid for GoF testing of semiparametric models by Genest and Rémillard (2008).

In their power study, Genest et al. (2009b) do not find any GoF test to be superior to the others under all circumstances. Performance and superiority of each test statistic depend on the combination of several factors like the level of dependence, the tested copula and the true underlying copula as well as sample size. The rank-based Cramér-von Mises test statistic S_n was selected for this study since S_n based on the empirical copula proved to be the most powerful test of the Clayton hypothesis and also did well for testing the Frank family in the power study of Genest et al. (2009b). Moreover, according to the reported average rankings of the test statistics in terms of performance, S_n was ranked second among all used test statistics. The test statistic ranked first is a Cramér-von Mises functional based on Rosenblatt's transform $S_n^{(B)}$. Genest et al. (2009b), however, point out that the difference might not be statistically significant and conclude that the rank-rank based S_n is one of the best blanket Goodness-of-Fit tests for copula models.

2.4 Simulating from Copulas

There are numerous methods of simulating from copulas. An in-depth coverage can, for example, be found in Mai and Scherer (2012). This section focuses on the Conditional Inversion Method for the bivariate case (see e.g. Cherubini et al., 2004, pp. 182). The extension for the multivariate setting is also given in Cherubini et al. (2004, pp. 182).
To simulate from a bivariate copula, we need to generate observations $(u, v) \in [0, 1]$ from the uniform distributed random variables $U = F_X(X)$ and $V = F_Y(Y)$ with joint distribution function C. The parameters of C are treated as known. For this purpose, the conditional distribution can be used given by the partial derivative of C

$$c_u(v) = \frac{\partial C}{\partial u} = \lim_{\Delta u \to 0} \frac{C(u + \Delta u, v) - C(u, v)}{\Delta u}$$

$$= \mathrm{P}(V \leq v \mid U = u) = \mathrm{P}(F_X(X) \leq v \mid F_Y(Y) = u),$$

where $c_u(v)$ is a non-decreasing function and exists for almost all $v \in [0, 1]$.
Using this result, the desired observational pairs (u, v) can be simulated in the following steps:

1. Sample independent and uniform $u, w \sim U[0, 1]$.
2. Use the (quasi-)inverse of $c_u(v)$ to obtain $v = c_u^{-1}(w)$.

The generated v have the desired distribution because

$$\mathrm{P}(F^{-1}(U) \leq x) = \mathrm{P}(U \leq F(x)) = F(x), \quad U \sim U[0, 1]$$

holds for any distribution function F, F^{-1} denoting the generalised inverse (see e.g Mai and Scherer, 2012, p. 234). If needed, the sample of (u, v) pairs can be transformed into a sample of (x, y) pairs using the respective inverses of the marginal distribution functions $F_X^{-1}(u)$ and $F_Y^{-1}(v)$.

2.5 GMM, ML, PML and MM

This section discusses the differences between the previously introduced estimation methods and puts them in a unifying framework for comparison. The purpose is to explain why it might make sense to resort to the Kendall's τ estimator instead of the ML estimator in case of a misspecified copula.

First of all, the term pseudo-maximum likelihood shall be discussed in detail as, in the copula context, it is used to denote various modifications of the true likelihood. In general, the term pseudo-maximum-likelihood is used to indicate that some modifications to the correct log-likelihood of FML in Equ. (12) are made. Roughly speaking, two kinds of modifications can be distinguished:

1. Unknown quantities are replaced by their estimates in a correctly specified model. This implies that the last term in Equ. (12) is omitted and the distribution functions in the first part are replaced by previously obtained estimates $\hat{F}$, either parametric or non-parametric. This gives IFM or CML, respectively.

2. Misspecification: H_0 in Subsection 2.2.1 does not hold. The assumed copula does not belong to the true parametric copula family, and thus, the copula density c in Equ. (12) is misspecified. Parametric misspecification of the marginals also induces a pseudo-likelihood optimisation. The optimisation of a misspecified likelihood gives the quasi-maximum likelihood estimator (QMLE), also referred to as pseudo-maximum likelihood estimator (PMLE).

The major difference between the two types of pseudo-likelihoods is that the optimisation of the first type gives estimates that converge to the true parameter, whereas the second does not in general. In the following, PML refers to the second case. The subsequent analysis compares the FML estimator to the inverted Kendall's τ estimator. Comparisons with IFM and CML are postponed to some final remarks.

To compare ML and the inverted Kendall's τ estimator, the concept of the Generalised Method of Moments (GMM) (Hansen, 1982) shall be recalled briefly. As many estimators can be seen as special cases of GMM, it serves as a unifying framework for comparison. The idea of GMM is to construct an estimator from exploiting the sample moment counterparts of population moment conditions, also referred to as the orthogonality conditions, of the true model. GMM refers to an over-identified system of equations, that is to say the number of moment equations k exceeds the number of unknown parameters p. It minimises a quadratic form of the sample analog of the population condition. If the system is exactly identified $k = p$, GMM reduces to the Method of Moments (MM) estimator. Provided that the model is correctly specified and some regularity conditions hold, GMM is consistent and asymptotically normal. For a detailed discussion of GMM, the reader is referred to Hall (2005) as an in-depth coverage of GMM is beyond the scope of this thesis.

ML can be interpreted as a case of GMM where the score vector is used to set up the moment conditions (see e.g. Hall, 2005). It belongs to the subgroup of M-estimators since it is the solution to a maximisation problem. The estimates $\hat{\alpha} = (\hat{\delta}_1, \ldots, \hat{\delta}_d, \hat{\theta})^\top$ obtained by FML in Subsection 2.2.1 solve the sample counterpart of the of the $d+1$ population moment conditions

$$\mathrm{E}[(\partial\ell/\partial\delta_1, \ldots, \partial\ell/\partial\delta_d, \partial\ell/\partial\theta)] = 0. \tag{21}$$

In contrast, using Kendall's τ gives a single moment condition that is solved for θ:

$$\mathrm{E}[\tau(\theta) - \hat{\tau}] = 0 \tag{22}$$

The inverted Kendall's τ estimator ranks among the MM estimators as one equation estimates one parameter.

As both estimators can be seen as special cases of the GMM estimator, let us first consider the GMM estimator under misspecification in general (Hall, 2005, pp. 117). Let the assumed bivariate copula model be denoted by C_0. A set of population moment conditions that may be used as a basis for the GMM estimator for the parameter θ is implied by C_0. This logical sequence may be presented as

$$C_0 \quad \Rightarrow \quad \mathrm{E}[f(u_i, v_i; \theta_0)] = 0, \quad \forall\, i,\ i \in \{1, \ldots, n\}, \text{ for a unique } \theta_0 \in \Theta, \tag{23}$$

where $f(\cdot)$ is a vector of functions. If C_0 is not the true data-generating copula, two scenarios are possible. The first scenario is that the property stated in Equ. (23) holds for the true model C_1, even though it is different from C_0, such that

$$C_1 \quad \Rightarrow \quad \mathrm{E}[f(u_i, v_i; \theta_+)] = 0, \quad \forall\, i,\ i \in \{1, \ldots, n\},\ \text{for a unique } \theta_+ \in \Theta. \tag{24}$$

The second possible scenario is that the true model C_2 does not share the property specified in Equ. (23)

$$C_2 \quad \Rightarrow \quad \nexists\, \theta \in \Theta \quad \text{such that } \mathrm{E}[f(u_i, v_i; \theta)] = 0, \quad \forall i\,, i \in \{1, \ldots, n\}. \tag{25}$$

In the first case, C_0 and C_1 are observationally equivalent on the basis of $\mathrm{E}[f(u_i, v_i; \theta)$. The estimator and the estimated sample moment have the same asymptotic properties under C_0 and C_1. This implies that the estimator is consistent and asymptotically normal in the first case. The only difference is in the use of θ_0 and θ_+ to denote the value at which the population moment condition and other regularity conditions are satisfied. As opposed to the first case, the implications of C_0 and C_2 are different. The second case of misspecification affects the behaviour of the estimator and the estimated sample moment.

The first scenario seems to apply in the motivating example if Kendall's τ is used for the calibration of a Clayton copula using Gumbel. Using MM gives unbiased results even under misspecification, which ML does not. Of course, it is important to be aware of the fact, that both models have totally different implications for the tail behaviour of the resulting distribution. So depending on the purpose and the respective application, using the wrong model for modelling and forecasting might yield misleading results.

Coming back to the mere purpose of estimating the copula dependence parameter, the fact that Gumbel and Clayton yield correct estimates for θ at a dependence level of $\tau = 0.5$ using the inverse of Kendall's τ comes not as a surprise. Figure 1 illustrates why. It plots the functional relationship between τ and θ for each of the three Archimedean copulas as stated in Table 2. Obviously, the $\tau(\theta)$-functions of the Gumbel and the Clayton copula intersect at $\tau = 0.5$ giving a θ of two for both copulas. In the neighbourhood of $\tau = 0.5$, the two functions are quite close to each other. For τ substantially greater or smaller than 0.5, the functions diverge more and more. However, compared to the Frank copula, the $\tau(\theta)$-functions of Gumbel and Clayton are rather close to each other over the whole interval of $[0, 0.75]$. They have a similar shape and the difference in the slope is not as significant as the difference in slope between the curves of Frank and Gumbel or Frank and Clayton. The curve for the Frank copula, in fact, is much steeper. The functions of Gumbel and Frank intersect at an

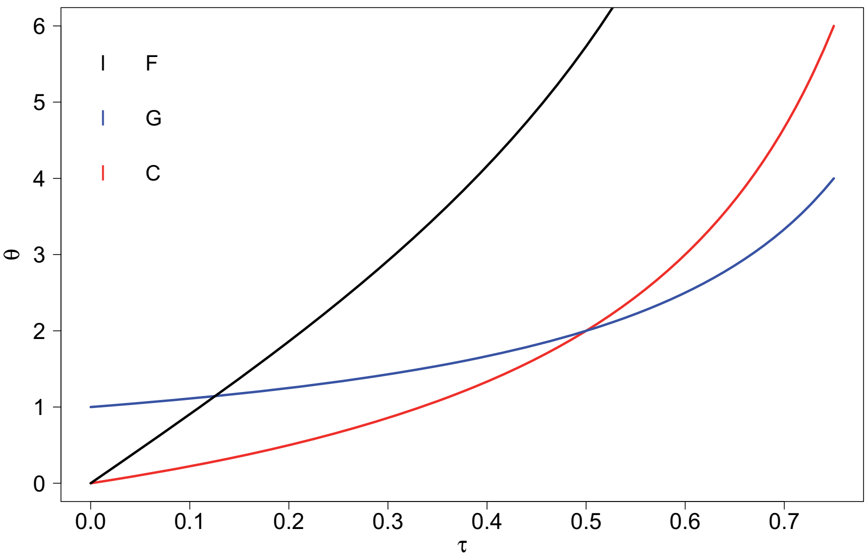

Figure 1: Copula dependence parameter θ as a function of τ over the interval $[0, 0.75]$ for the Frank, the Gumbel and the Clayton copula.

approximate τ of 0.1255. The $\tau(\theta)$-functions of the Clayton and the Frank copula only share a common point for $\tau = 0$, which is not of interest as for $\theta = 0$ the independence copula results.

All in all, the $\tau(\theta)$-function of one copula serves as a good approximation for the $\tau(\theta)$-function of another copula for certain levels of dependence. Hence, the first scenario applies in some cases using Kendall's τ and good estimates of θ can be obtained even under copula misspecification.

Under the assumption that the model is correctly specified, and hence, $\ell(\alpha; y_1, \ldots, y_n)$ in Equ. (12) is the correct log-likelihood, it follows that the MLE has the optimal properties of consistency and asymptotic efficiency and constitutes the preferred first option. Again, provided that the correct model has been identified, ML is the best option. However, if the model is misspecified, and thus $\ell(\cdot)$ is not the correct log-likelihood, the maximiser of $\ell(\cdot)$ is not the MLE. FML loses its preferable properties and status.

The quality of ML under misspecification also depends on how well the assumed density approximates the true one. Optimisation of the misspecified log-likelihood gives the PML / QML estimator which converges to the pseudo-true value. The pseudo-true value minimises the Kullback-Leibler divergence of the assumed copula density relative to the true density (for a formal treatment see e.g. Joe, 2014, pp. 227). However, even though the distance is minimised, it might still be substantial and the pseudo-true value might significantly differ from the true parameter value. QML will only yield reasonable results in the rather rare case that the assumed density is a very good approximation of the true one. If the copula function is incorrect, the joint distribution is misspecified. This generally means that estimators based on the joint likelihood will be inconsistent. In particular, the copula dependence parameter will be estimated inconsistently (Prokhorov, 2008). The results of the motivating example also point to the fact that ML gives incorrect results under misspecification. As such, in general, the second scenario is more likely to apply for ML under copula misspecification.

The preceding analysis compared ML and the inversion of Kendall's τ. The comparison between IFM, CML and the inversion of Kendall's τ is confined to some rough considerations. For a correctly specified model, IFM is also quite efficient compared to FML and may as such be preferable (Joe, 1997, 2005) to the inversion of Kendall's τ. For CML no statements about efficiency can be made in general. Inconsistency, however, also is a consequence of copula misspecification for IFM and CML (Prokhorov, 2008). So again, if the $\tau(\theta)$-functions of the true and the assumed copula approximate each other well, the inverted Kendall's τ might be more likely to yield good results.

Recall the proposed estimator from Equ. (1)

$$\hat{\theta}_{new} = p \cdot \hat{\theta}_{ML} + (1-p) \cdot \hat{\theta}_{\tau}.$$

In the light of the preceding discussion, it seems reasonable to choose the inverse of Kendall's τ if the copula used for estimation is likely to be misspecified. This selection is "automated" using the p-value weighted average of the two estimators. The less likely it is that the assumed distribution is in fact the true one, the less weight is attributed to the ML estimator and the more weight is attached to the inverse of Kendall's τ. Conversely, if the assumed distribution is very likely to be the true one, implying the GoF test returns a p-value close to one, the resulting $\hat{\theta}_{new}$ corresponds to the ML estimator. As such, the proposed estimator is expected to place most weight on the most preferable estimate. To check this presumption and to assess the performance of the proposed estimator, a simulation study was conducted (see Section 4) and the estimator was tried with real data (see Section 5).

3 Estimation of the Value-at-Risk Using Copulas

Due to the non-Gaussian behaviour of returns manifested in fat tails and asymmetry, a frequent application of copulas is the estimation of the Value-at-Risk (VaR) of a portfolio comprising two or more assets. This section provides the theoretical background for the empirical example in Section 5, where the proposed estimator will be compared to the standard ML and the Kendall's τ estimator in terms of VaR performance evaluation through backtesting. The main assumptions and steps for the dynamic estimation of the VaR from a Profit-and-Loss function of a linear portfolio are illustrated, see Härdle and Okhrin (2010) and Giacomini et al. (2009). The discussion will be limited to the bivariate case.

Let $w = (w_1, w_2)^\top \in \mathbb{R}^2$ denote a portfolio consisting of two positions in two assets and let the non-negative random vector of prices of the assets at time t be represented by $S_t = (S_{1,t}, S_{2,t})^\top$. The value V_t of the portfolio w is then defined by

$$V_t = w_1 S_{1,t} + w_2 S_{2,t}.$$

The random variable

$$L_t = (V_t - V_{t-1}), \quad S_{1,0} = S_{2,0} = 0 \tag{26}$$

is called the profit and loss (P&L) function. It equals the change in the portfolio value between two subsequent time points. Using log-returns $X_{j,t} = \log S_{j,t} - \log S_{j,t-1},\ j = 1, 2,$ the P&L function in Equ. (26) can be reformulated

$$L_t = w_1\ S_{1,t-1}\{\exp(X_{1,t} - 1)\} + w_2\ S_{2,t-1}\{\exp(X_{2,t}) - 1\}. \tag{27}$$

The distribution function of L_t is given by

$$F_{t,L_t}(x) = P_t(L_t \le x). \tag{28}$$

The Value-at-Risk at level α from a portfolio w is defined as the α-quantile from F_{t,L_t}:

$$\mathrm{VaR}_t(\alpha) = F^{-1}_{t,L_t}(\alpha). \tag{29}$$

It follows from Equ. (28) and Equ. (29) that F_{t,L_t} depends on the two-dimensional distribution of log-returns X_t. Therefore, to derive the quantiles in Equ. (29), modelling the distribution of log-returns is crucial to the estimation of the VaR. As changes in the log-returns basically drive the change in the portfolio value, log-returns constitute a risk factor.

Generally speaking, all factors driving the portfolio value are referred to as risk factors. More specifically, changes in the underlying risk factors that influence the P&L of a portfolio govern the loss distribution F_{t,L_t}. There is a variety of risk factors that can be used for risk modelling such as foreign exchange rates, commodity prices, interest rates or volatility to name just a few examples.

For the empirical example in Section 5, log-returns represent a suitable and sufficient risk factor choice. Log-returns are modelled over time via the process $\{X_t\}$

$$X_{j,t} = \mu_{j,t} + \sigma_{j,t}\epsilon_{j,t}, \quad j = 1, 2$$

where $\epsilon_t = (\epsilon_{1,t}, \epsilon_{2,t})^\top$ are standardised i.i.d. innovations with $\mathrm{E}[\epsilon_{j,t}] = 0$ and $\mathrm{E}[\epsilon_{j,t}^2] = 1$ for $j = 1, 2$. Let $\mathcal{F}_t$ denote the information set at time t.

$$\mu_{j,t} = \mathrm{E}[X_{j,t} \mid \mathcal{F}_{t-1}]$$

is the conditional mean given the information set of the preceding time interval $\mathcal{F}_{t-1}$, and

$$\sigma_{j,t}^2 = \mathrm{E}[(X_{j,t} - \mu_{j,t})^2 \mid \mathcal{F}_{t-1}]$$

is the conditional variance given $\mathcal{F}_{t-1}$. The joint distribution function of the innovations $\epsilon_t = (\epsilon_{1,t}, \epsilon_{2,t})^\top$ is

$$F_\epsilon(\epsilon_1, \epsilon_2) = C_\theta\{F_1(\epsilon_1), F_2(\epsilon_2)\}, \tag{30}$$

where C_θ denotes the assumed copula of the parametric family $\mathcal{C} = \{C_\theta,\ \theta \in \Theta\}$, and F_1, F_2 are the continuous marginal distributions of ϵ_j.

To obtain the Value-at-Risk in this set-up, several steps are needed. First, the residuals are estimated from the sample of log-returns. Then, to obtain an estimate of F_ϵ, the estimated residuals are used to estimate the copula dependence parameter and the parameters of the marginal distribution functions of the residuals in Equ. (30). Finally, $\hat{F}_\epsilon$ is employed to generate P&L Monte Carlo samples whose quantiles at different levels serve as estimators for the Value-at-Risk.

The procedure of estimating the Value-at-Risk at level α for a portfolio w consisting of two assets and a sample $\{x_{j,t}\}_{t=1}^T$, $j = 1, 2$ of log-returns can be summarised in five steps:

1. Estimation of the residuals $\{\hat{\epsilon}_t\}_{t=1}^T$ using a pre-specified time-series model, e.g. GARCH.

2. Specification and estimation of the marginal distributions $F_j(\hat{\epsilon}_j)$.

3. Specification of a parametric copula family $\mathcal{C}$ and estimation of the dependence parameter θ.

4. Generation of Monte Carlo sample of innovations ϵ and losses L for the forecast on that one day.

5. Estimation of $\widehat{\mathrm{VaR}}_t(\alpha)$, the empirical α-quantile of F_L.

These steps give a single Value-at-Risk estimate $\widehat{\mathrm{VaR}}_t(\alpha)$. In the dynamic approach, the procedure is applied on moving windows of the time series $\{x_{j,t}\}_{t=1}^T$, $j = 1, 2$. Using moving windows of size r in time t

$$\{x_t\}_{t=s-r+1}^s$$

for $s = r, \ldots, T$ generates the time series $\{\widehat{\mathrm{VaR}}_t(\alpha)\}_{t=r}^T$ of Value-at-Risk estimates and $\{\hat{\theta}\}_{t=r}^T$ dependence parameter estimates.

By means of backtesting, the performance of the copula, and more importantly in this study, the performance of the respective estimator for the copula dependence parameter is evaluated. To that end, the VaR estimates $\{\widehat{\mathrm{VaR}}_t(\alpha)\}_{t=r}^T$ are compared to the true realisations $\{l_t\}$ of the P&L function and the exceedances ratio $\hat{\alpha}$ is computed

$$\hat{\alpha} = \frac{1}{T-r} \sum_{t=r}^{T} \mathbf{I}\{l_t < \widehat{\mathrm{VaR}}_t(\alpha)\}. \tag{31}$$

The realised α is estimated via $\hat{\alpha}$ and should approximately be equal to the theoretical α. The relative difference e between the theoretical α and the empirical $\hat{\alpha}$ is calculated by

$$e = |\hat{\alpha} - \alpha| / \alpha. \tag{32}$$

4 Simulation Study

The performance of the proposed estimator is assessed by means of a simulation study. Two sample sizes are studied to examine small and large sample performance. The study is limited to the bivariate case considering the three well-known Archimedean copula families Frank, Gumbel, and Clayton.

In the first part, three standard levels of dependence are studied corresponding to Kendall's $\tau \in \{0.25, 0.5, 0.75\}$. For each possible combination of copula and fixed value of τ, 1000 random samples of size 500 as well as 10 000 random samples of size 50 are simulated. To each data set the Frank, the Gumbel and the Clayton copula are fitted and an estimate for θ is obtained using maximum likelihood, the inversion of Kendall's τ and by forming a p-value weighted average of the two (see Equ. (1)). In case the sampling copula and the copula used for estimation do not coincide, copula misspecification occurs. In this case, the MLE turns into the PMLE. For convenience and as the purpose is to generally compare different estimation approaches, both cases will be referred to as ML. ML shall be understood as a generic term in this case that comprises PML.

The inversion of Kendall's τ is expected to render correct results for τ's close to the τ where the $\tau(\theta)$-functions of two copulas intersect, even if sampling copula and estimation copula differ. The three standard dependence levels $\tau \in \{0.25, 0.5, 0.75\}$ include the dependence level at which the $\tau(\theta)$-functions of Gumbel and Clayton intersect. As shown in Figure 1, the $\tau(\theta)$-functions of the Frank and Gumbel copula intersect at an approximate τ of 0.1255. Therefore, this dependence level is also investigated for Frank and Gumbel.

In the second part, samples from mixtures of Gumbel and Clayton for several $\alpha's$, according to Equ. (4), are considered for the dependence level $\tau = 0.5$. Why only this dependence level is taken into account, will become evident from the results of the first part of the simulation study. Calculations in the second part are based on 5000 random samples of size $n = 100$.

All computations were done using the statistical software R, in particular, the *copula* package (Kojadinovic and Yan, 2010b). For simulation, the conditional approach is used, which is also implemented in R software. Without loss of generality, the margins of the copula are assumed to be uniform on $[0, 1]$. Misspecification of the marginals is left aside in this study.

Results are presented in Tables 3, 4, 6 and 7 and Figures 2 to 4. For a given estimation method, let $\hat{\theta}^{(i)}$ denote the estimator of θ for the ith repeated sample, let N denote the number of random samples, and n the sample size. If not stated otherwise, all subsequent sums run over i from one to N. Limits are omitted for notational convenience. The following coefficients shall be used to assess the performance of each estimator:

1. estimated mean: $\bar{\theta} = N^{-1} \sum \hat{\theta}^{(i)}$
2. estimated standard deviation of obtained estimates: $\{(N-1)^{-1} \sum (\hat{\theta}^{(i)} - \bar{\theta})^2\}^{-1/2}$
3. estimated bias: $N^{-1} \sum (\hat{\theta}^{(i)} - \theta)$
4. estimated Percentage Relative Bias: $\mathrm{PRB} = N^{-1} \sum \hat{\theta}^{(i)}/\theta$
5. mean of estimated Kullback-Leibler divergence between true and estimated model: $\overline{\mathrm{KL}}(c_0, c_1) = N^{-1}[n^{-1} \sum c_0(u_i, v_i) \log\{c_0(u_i, v_i)/c_1(u_i, v_i)\}]$[1], where c_0 is the true copula density.
6. estimated mean squared error $\widehat{\mathrm{MSE}} = N^{-1} \sum (\hat{\theta}^{(i)} - \theta)^2$
7. estimated MSE efficiency of a given estimator relative to $\hat{\theta}_{new}$, $\{\widehat{\mathrm{MSE}}_{given\ estimator}\}\{\widehat{\mathrm{MSE}}_{\hat{\theta}_{new}}\}^{-1}$

As some tables are lengthy and hard to get a quick grasp of, some of the results are presented with the aid of figures instead. The corresponding tables are attached in the Appendix. In the following analysis as well as in the subsequent tables and figures, the Frank, the Gumbel, and the Clayton copula will be abbreviated by either F, G, and C or by Fra, Gum, and Clay respectively. Moreover, TC stands for True Copula and EC for Estimation Copula. Clearly, if EC does not correspond to TC, the dependence parameter θ is estimated under misspecification. Combinations of TC and EC will be abbreviated by TC-EC, meaning the true copula is stated first, followed by the copula used for estimation. For example, Fra-Gum indicates that θ was estimated using a sample from Frank and assuming a Gumbel copula.

[1]Note: Here, the sum runs from 1 to n.

TC	EC	$\hat{\theta}_{ML}$			$\hat{\theta}_{\tau}$			$\hat{\theta}_{new}$			p	$\hat{\tau}$	t
		$\bar{\theta}$	KL	Bias	$\bar{\theta}$	KL	Bias	$\bar{\theta}$	KL	Bias			(in s)
Fra	Fra	2.38	-0.00	0.01	2.38	-0.00	0.01	2.38	-0.00	0.01	0.49	0.25	115.3
($\theta = 2.37$)		0.29	0.02		0.29	0.02		0.29	0.02		0.29	0.03	1.8
	Gum	1.27	0.04	-1.10	1.34	0.02	-1.04	1.33	0.02	-1.04	0.02		245.1
		0.04	0.02		0.05	0.02		0.05	0.02		0.06		6.9
	Clay	0.44	0.07	-1.94	0.67	0.05	-1.70	0.67	0.05	-1.70	0.00		108.1
		0.07	0.02		0.10	0.02		0.10	0.02		0.02		3.0
Gum	Fra	2.42	2.16	1.08	2.38	2.17	1.05	2.38	2.16	1.05	0.05	0.25	115.5
($\theta = 1.33$)		0.31	16.19		0.31	16.22		0.31	16.22		0.11	0.03	1.7
	Gum	1.34	-0.01	0.01	1.34	0.00	0.00	1.34	-0.00	0.01	0.50		245.1
		0.05	0.19		0.05	0.22		0.05	0.20		0.28		4.9
	Clay	0.41	2.51	-0.93	0.67	2.41	-0.66	0.67	2.41	-0.66	0.00		109.6
		0.07	17.27		0.10	16.99		0.10	16.99		0.00		4.3
Clay	Fra	2.40	3.52	1.73	2.37	3.53	1.70	2.37	3.53	1.70	0.01	0.25	138.2
($\theta = 0.67$)		0.30	38.86		0.30	38.87		0.30	38.87		0.03	0.03	6.8
	Gum	1.26	2.93	0.60	1.33	2.68	0.67	1.33	2.68	0.67	0.00		291.7
		0.04	30.83		0.05	28.22		0.05	28.22		0.00		13.1
	Clay	0.68	-0.02	0.02	0.67	0.01	0.00	0.68	-0.01	0.01	0.50		126.6
		0.09	0.26		0.10	0.31		0.10	0.28		0.29		7.3
Fra	Fra	5.74	-0.00	0.00	5.74	-0.00	0.00	5.74	-0.00	0.00	0.51	0.50	112.8
($\theta = 5.74$)		0.36	0.03		0.37	0.03		0.36	0.03		0.29	0.02	1.5
	Gum	1.78	0.16	-3.95	2.00	0.11	-3.73	2.00	0.11	-3.73	0.00		213.7
		0.06	0.04		0.08	0.05		0.08	0.05		0.00		7.1
	Clay	1.12	0.34	-4.62	2.00	0.33	-3.73	2.00	0.33	-3.73	0.00		112.2
		0.09	0.05		0.16	0.09		0.16	0.09		0.00		1.8
Gum	Fra	5.74	11.99	3.74	5.75	12.00	3.75	5.75	12.00	3.75	0.00	0.50	134.6
($\theta = 2.0$)		0.40	68.41		0.41	68.45		0.41	68.45		0.01	0.02	6.4
	Gum	2.01	-0.01	0.01	2.00	0.01	0.00	2.01	0.00	0.01	0.51		243.0
		0.09	0.47		0.09	0.58		0.09	0.52		0.29		11.7
	Clay	1.09	14.36	-0.91	2.01	13.58	0.01	2.01	13.59	0.01	0.00		134.1
		0.11	75.97		0.18	73.33		0.18	73.33		0.00		8.5
Clay	Fra	5.70	27.39	3.70	5.75	27.36	3.75	5.75	27.36	3.75	0.00	0.50	134.3
($\theta = 2.0$)		0.42	316.61		0.43	316.51		0.43	316.51		0.00	0.02	6.2
	Gum	1.74	17.55	-0.26	2.00	15.00	0.00	2.00	15.00	0.00	0.00		256.9
		0.07	186.08		0.10	159.92		0.10	159.93		0.00		14.2
	Clay	2.01	0.01	0.01	2.01	-0.01	0.01	2.02	-0.02	0.02	0.49		129.4
		0.18	1.13		0.19	0.81		0.18	0.80		0.29		7.5
Fra	Fra	14.15	-0.00	0.01	14.21	-0.01	0.07	14.18	-0.00	0.04	0.50	0.75	115.1
($\theta = 14.14$)		0.66	0.06		0.67	0.06		0.66	0.06		0.29	0.01	1.8
	Gum	3.11	0.71	-11.03	4.02	0.72	-10.12	4.02	0.72	-10.12	0.00		216.0
		0.12	0.17		0.16	0.25		0.16	0.25		0.00		3.3
	Clay	2.77	1.48	-11.37	6.03	2.14	-8.10	6.03	2.14	-8.11	0.00		101.8
		0.22	0.21		0.33	0.55		0.33	0.55		0.00		2.1
Gum	Fra	13.83	28.53	9.83	14.20	28.36	10.20	14.19	28.36	10.19	0.00	0.75	136.3
($\theta = 4.0$)		0.85	125.14		0.88	124.75		0.88	124.75		0.00	0.01	6.0
	Gum	4.01	-0.03	0.01	4.01	-0.01	0.01	4.02	-0.02	0.02	0.49		232.1
		0.20	0.77		0.22	0.69		0.21	0.75		0.29		11.2
	Clay	2.95	34.27	-1.05	6.03	33.47	2.03	6.03	33.47	2.03	0.00		120.8
		0.25	138.87		0.43	134.59		0.43	134.59		0.00		6.9
Clay	Fra	13.75	1061.89	7.75	14.24	1059.84	8.24	14.24	1059.84	8.24	0.00	0.75	136.7
($\theta = 6.0$)		0.87	28857.51		0.90	28812.52		0.90	28812.54		0.00	0.01	6.2
	Gum	2.85	419.14	-3.15	4.03	324.59	-1.97	4.02	324.63	-1.98	0.00		258.4
		0.14	11153.25		0.22	8719.45		0.22	8720.32		0.00		11.5
	Clay	5.96	2.15	-0.04	6.05	-2.06	0.05	6.05	-1.66	0.05	0.50		121.5
		0.41	54.03		0.44	74.15		0.42	64.87		0.29		6.8

Table 3: The columns contain the sample average of obtained estimates ($\bar{\theta}$), the average of the estimated Kullback-Leibler divergence between the estimated and the true model (KL), and the average bias for the maximum likelihood estimator $\hat{\theta}_{ML}$, the inverted Kendall's τ ($\hat{\theta}_{\tau}$), and for the p-value weighted average of the two ($\hat{\theta}_{new}$), respectively. 1000 data sets of size $n = 500$ were sampled from the Frank (Fra), the Gumbel (Gum), and the Clayton (Clay) copula for $\tau = 0.25$ (*upper part*), $\tau = 0.5$ (*middle part*), and $\tau = 0.75$ (*lower part*). Second lines contain the corresponding standard deviations. COPSimStudyResLarge

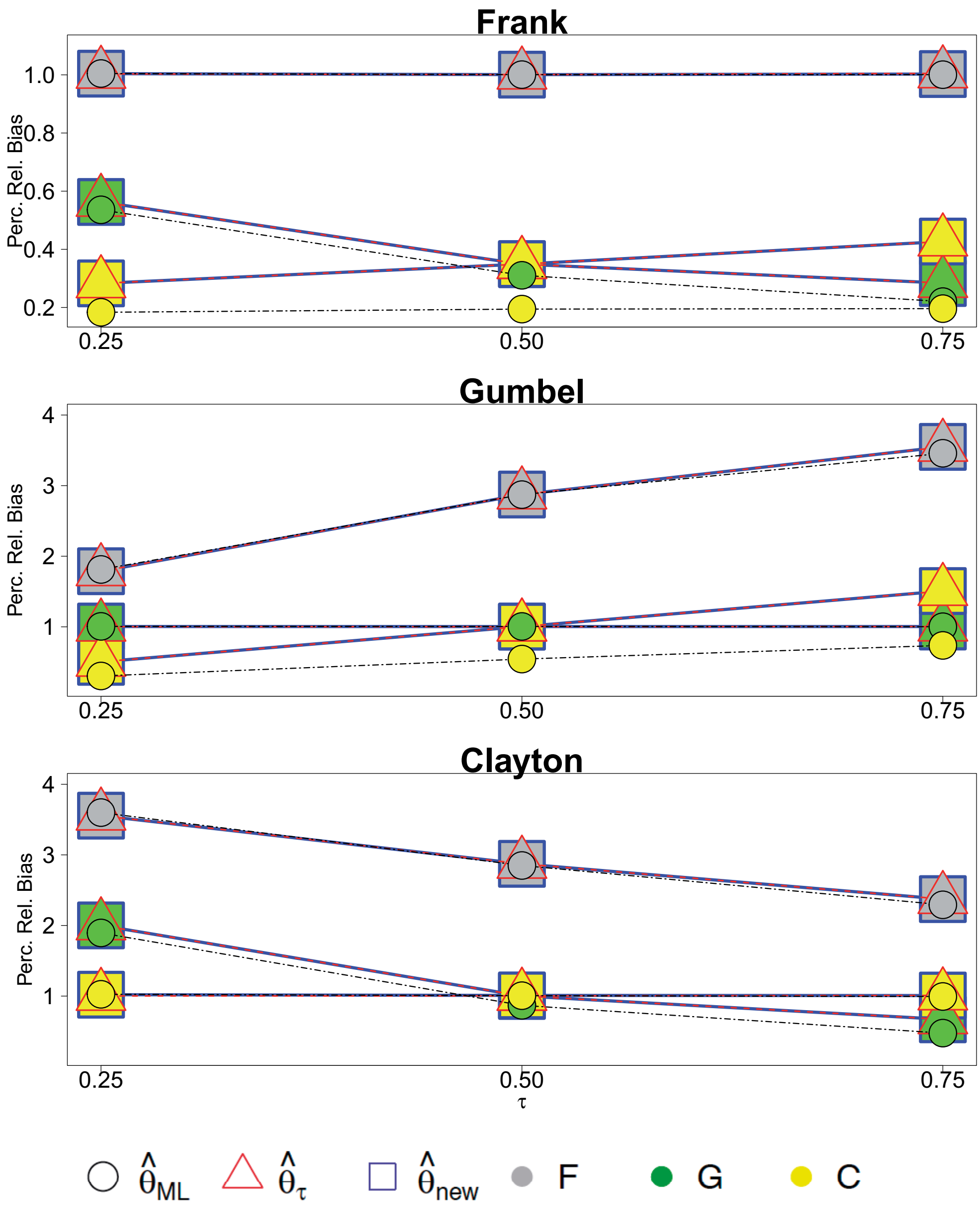

Figure 2: The figure plots the estimated percentage relative bias for each combination of estimator and estimation copula as function of the dependence level τ for a sample size of $n = 500$. The estimator is indicated by the line colour and the shape, while the estimation copula is represented by the fill colour of each shape. The titles of the plots state the true copula. F, G and C stand for Frank, Gumbel and Clayton, respectively.

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4.1 Non-Mixture Samples

Figure 2 plots the estimated percentage relative bias (PRB) for each estimator in combination with each copula as a function of the dependence level τ. Each plot presents the results for one of the three Archimedean copulas being the true copula. It is apparent that if the true copula is used for estimation, the three estimators coincide and give unbiased results for all three dependence levels. This corresponds to an estimated PRB of one for all dependence levels and estimators. Moreover, it is obvious that estimates obtained by the Gumbel and the Clayton copula are closer and estimates obtained using the Frank copula are substantially different, independent of the estimation method. Gumbel and Clayton significantly underestimate the true dependence parameter if the observations come from a Frank copula. Conversely, Frank severely overestimates the true parameter if the sample originates from Gumbel or Clayton. For $\tau = 0.5$, using Gumbel and Clayton in combination with $\hat{\theta}_{new}$ or $\hat{\theta}_{\tau}$ gives unbiased results no matter which of the two copulas is the true one. Put differently, $\hat{\theta}_{new}$ or $\hat{\theta}_{\tau}$ are robust against copula misspecification at $\tau = 0.5$ if the sample comes from Gumbel or Clayton and the respective other copula is used for estimation. An explanation was already provided in Subsection 2.5 in Figure 1.

Table 3 additionally admits the following observations:

1. In case a misspecified copula is used, the GoF test correctly detects the misspecification and yields p-values very close to zero. As such, $\hat{\theta}_{new}$ almost gives the same results as $\hat{\theta}_{\tau}$.

2. The proposed estimator yields better results if either the Gumbel or the Clayton copula are used for sampling and/ or for estimation compared to employing the Frank copula for either of these purposes.

3. If the Frank copula is used as the misspecified copula, the difference between $\hat{\theta}_{ML}$ and $\hat{\theta}_{\tau}$ only is notable at a dependence level of $\tau = 0.75$. However, since $\hat{\theta}_{\tau}$ is more biased on average in this case, $\hat{\theta}_{new}$ does not succeed in selecting the estimate with the least bias. For the two lower dependence levels, the average bias of the ML and the MM estimator is approximately the same. Therefore, for Frank as EC, $\hat{\theta}_{new}$ cannot add any value.

4. If Gumbel and Clayton are erroneously used for estimation, $\hat{\theta}_{ML}$ returns more severely biased estimates than $\hat{\theta}_{\tau}$. The only exceptions occur at $\tau = 0.25$ for Clay-Gum and at $\tau = 0.75$ for Gum-Clay.

5. Beside the two above-mentioned exceptions, $\hat{\theta}_{new}$ successfully selects the less biased estimate $\hat{\theta}_{\tau}$, if the Clayton or the Gumbel copula are used as misspecified copula in the estimation.

6. If Gumbel or Clayton are used as misspecified copula, the estimated KL distance between the true and the estimated model is in almost all cases of misspecification smaller

True copula	Estimation copula	$\tau = 0.25$		$\tau = 0.5$		$\tau = 0.75$	
		$\hat{\theta}_{ML}/\hat{\theta}_{new}$	$\hat{\theta}_{\tau}/\hat{\theta}_{new}$	$\hat{\theta}_{ML}/\hat{\theta}_{new}$	$\hat{\theta}_{\tau}/\hat{\theta}_{new}$	$\hat{\theta}_{ML}/\hat{\theta}_{new}$	$\hat{\theta}_{\tau}/\hat{\theta}_{new}$
Frank	Frank	1.00	1.00	0.99	1.01	1.00	1.03
	Gumbel	1.13	1.00	1.12	1.00	1.19	1.00
	Clayton	1.29	1.00	1.53	1.00	1.96	1.00
Gumbel	Frank	1.06	1.00	1.00	1.00	0.93	1.00
	Gumbel	1.02	1.01	1.01	1.04	0.93	1.07
	Clayton	1.93	1.00	25.81	1.00	0.27	1.00
Clayton	Frank	1.03	1.00	0.97	1.00	0.89	1.00
	Gumbel	0.80	1.00	8.01	1.00	2.52	1.00
	Clayton	0.89	1.08	0.94	1.08	0.96	1.11

Table 4: The columns present the estimated MSE efficieny of the ML and the Kendall's τ estimator relative to the proposed estimator for all dependence levels.

for $\hat{\theta}_{new}$ and $\hat{\theta}_{\tau}$ than for $\hat{\theta}_{ML}$. For Frank, KL distances are about the same for all three estimators.

7. KL distances shoot up under misspecification for Gumbel and Clayton as TC, especially at $\tau = 0.75$ and Clayton as TC. Standard deviations of the KL distances likewise blow up. In contrast, for Frank as TC, estimated KL distances in case of misspecification are rather moderate for all three dependence levels.

8. For $\tau = 0.5$, the proposed estimator yields correct estimates for θ even for Gum-Clay and Clay-Gum. This result is illustrated in Figure 3.

9. Note that even though Gumbel and Clayton give correct estimates for θ using $\hat{\theta}_{new}$ and $\hat{\theta}_{\tau}$ in case of mutual misspecification, the estimated KL distance is still remarkable.

10. If the true copula is used for estimation, $\hat{\theta}_{ML}$ and $\hat{\theta}_{\tau}$ are very close, so computing the p-value weighted average of the two more or less gives the same result. Moreover, in this case, average biases of the estimates are negligible for all three estimators.

Analysing the estimated MSE efficiency in Table 4 further allows the following conclusions:

1. If the true copula is used for estimation, the relative MSE are close to one in most cases implying there is not much to lose applying $\hat{\theta}_{new}$. The only exception constitutes the Clayton copula, especially for $\tau = 0.25$.

2. Under misspecification, some relative MSE are significantly greater than one indicating that $\hat{\theta}_{new}$ performs considerably better in terms of MSE than the respective estimator

it is compared to. Note that for $\tau = 0.5$ the relative MSE shoots up to 25.81 for the combination Gum-Clay. For the same level of dependence, the relative MSE is 8.01 if Clayton is used for simulation and Gumbel for estimation (Clay-Gum).

3. The only cases where $\hat{\theta}_{new}$ performs substantially worse than the respective estimator in terms of MSE are the before mentioned two exceptions, where $\hat{\theta}_{\tau}$ is more biased than $\hat{\theta}_{ML}$ for Clay-Gum at $\tau = 0.25$ and for Gum-Clay at $\tau = 0.75$.

4. Likewise, at a dependence level of $\tau = 0.75$, ML is preferable compared to $\hat{\theta}_{new}$ in terms of MSE if the Frank copula is erroneously used for estimation.

5. For all other combinations, the relative MSE is close to one or even greater, sometimes even substantially greater, than one, implying there is not much to lose but only to win using $\hat{\theta}_{new}$.

The results obtained using a sample size of $n = 50$ and doing 10 000 replications, confirm the above observations. The same conclusions can be drawn from Table 19 and Appendices B.2 and B.2. For brevity, the Tables are not discussed in detail and are attached in the appendix. The only point that shall be remarked is that, in contrast to the high MSE efficiency of $\hat{\theta}_{new}$ at $\tau = 0.5$ for Clay-Gum relative to $\hat{\theta}_{ML}$, the estimated relative MSE efficiency is smaller than one for the small samples indicating that ML is superior in terms of MSE. This result comes from the relative small standard error of $\hat{\theta}_{ML}$ relative to $\hat{\theta}_{new}$, not because $\hat{\theta}_{ML}$ is less biased. In fact, $\hat{\theta}_{new}$ gives an almost unbiased result, which ML does not.

In contrast to the large samples, the GoF test has some difficulties to detect the misspecification. The rather high p-values entail that the influence of the ML-estimate is higher than in the large sample case. In case Gumbel or Clayton are used for estimation, this even has a slightly positive effect. As ML underestimates the true parameter and the inversion of Kendall's τ slightly overestimates it, the average bias of $\hat{\theta}_{new}$ is sometimes even slightly lower than that of $\hat{\theta}_{\tau}$. For Frank as EC, the ML and MM estimates tend to be closer together, so the weighting is less relevant. Estimated PRB are also very similar to the ones obtained for the large samples as shown in Figure 10 in the Appendix.

Correct estimates are obtained under misspecification for Gum-Clay and Clay-Gum at the dependence level $\tau = 0.5$ for both sample sizes. Figure 3 illustrates the distribution of the estimates obtained by $\hat{\theta}_{ML}$, $\hat{\theta}_{\tau}$, and $\hat{\theta}_{new}$ in this case. For both sample sizes and both cases of misspecification, Gum-Clay and Clay-Gum, ML underestimates the true parameter. In contrast, $\hat{\theta}_{\tau}$ and $\hat{\theta}_{new}$ and give unbiased results even under misspecification. Interestingly, the

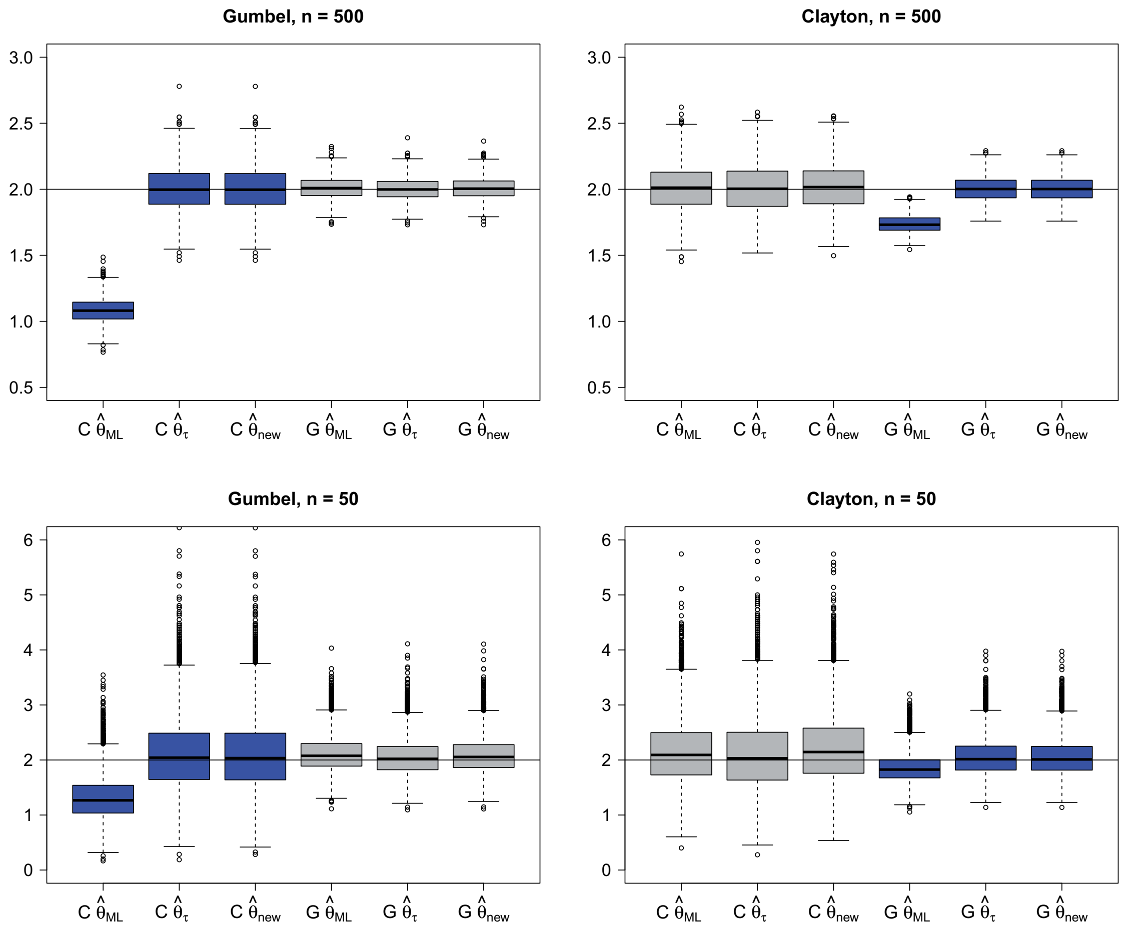

Figure 3: Distribution of the estimates obtained by $\hat{\theta}_{ML}$, $\hat{\theta}_{\tau}$ and $\hat{\theta}_{new}$ for $\tau = 0.5$ using the Gumbel and Clayton copula. Blue boxplots indicate that estimates were obtained with a misspecified copula. Grey boxplots show the distribution of estimates obtained by the true copula. The upper panel shows the large sample results, the lower the small sample results.

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dispersion of estimates around the mean is smaller for Clay-Gum (right panel, blue boxplots) than for Clay-Clay. Put differently, using Gumbel instead of the true copula Clayton not only gives unbiased results for $\hat{\theta}_{\tau}$ and $\hat{\theta}_{new}$, but also more stable results. If the sample comes from Gumbel and Clayton is used for estimation, this is, however, not the case. The dispersion of estimates around the mean is greater if the misspecified copula is used. For the small samples, $\hat{\theta}_{ML}$ and $\hat{\theta}_{new}$ slightly overestimate the true parameter if the true copula is used. In contrast, under misspecification, $\hat{\theta}_{\tau}$ and $\hat{\theta}_{new}$ yield correct results.

As the $\tau(\theta)$-functions of Gumbel and Frank intersect at $\tau \approx 0.1255$ (see Figure 1) and the proposed estimator proved to work very well for Clayton and Gumbel at their intersection point of $\tau = 0.5$, simulations involving the Gumbel and the Frank copula were also done for $\tau = 0.1255$. 1000 random samples of size $n = 500$ were generated from each copula and θ was

TC	EC	$\hat{\theta}_{ML}$			$\hat{\theta}_{\tau}$			$\hat{\theta}_{new}$			p	$\hat{\tau}$	t
		$\bar{\theta}$	KL	Bias	$\bar{\theta}$	KL	Bias	$\bar{\theta}$	KL	Bias			(in s)
Fra	Fra	1.15	-0.00	0.00	1.15	-0.00	0.00	1.15	-0.00	0.00	0.50	0.13	101.2
		0.27	0.01		0.27	0.01		0.27	0.01		0.29	0.03	14.6
	Gum	1.11	0.01	-0.03	1.14	0.01	0.00	1.14	0.01	-0.00	0.14		234.4
		0.03	0.01		0.04	0.01		0.04	0.01		0.19		36.8
Gum	Fra	1.17	0.60	0.03	1.16	0.60	0.01	1.16	0.60	0.02	0.23	0.13	125.2
		0.28	3.83		0.27	3.83		0.28	3.83		0.25	0.03	14.5
	Gum	1.15	-0.01	0.01	1.15	0.01	0.00	1.15	0.00	0.01	0.51		281.3
		0.04	0.11		0.04	0.15		0.04	0.13		0.29		32.6

Table 5: The columns contain the sample average of obtained estimates ($\bar{\theta}$), the average of the estimated Kullback-Leibler divergence between the estimated and the true model (KL), and the average bias for the maximum likelihood estimator $\hat{\theta}_{ML}$, the inverted Kendall's τ ($\hat{\theta}_{\tau}$), and for the p-value weighted average of the two, respectively. 1000 data sets of size $n = 500$ were sampled from the Frank (Fra) and the Gumbel (Gum) copula for $\tau = 0.1255$. Second lines contain the corresponding standard deviations.

estimated using the Frank and the Gumbel copula coupled with each of the three estimators. In this instance, calculations were only run for the large sample size $n = 500$, as at such a low level of dependence, the GoF test cannot cope with a smaller sample size.

It turns out at that, at such a low level of dependence, even a misspecified likelihood, meaning the PML estimator, yields not too severely biased results. The pseudo-true value and the true value are very close resulting in biases of very small magnitude as well as small percentage relative biases (for PRB see Table 20 in the Appendix). Apparently, the two copula densities are rather close and a good approximation of each other at such a low level of dependence.

The GoF test has some difficulties to detect the misspecified copula and p-values are rather high for the erroneously used copula. Even though average biases are small for ML in case of misspecification, the average bias of the inverted Kendall's τ is still smaller. The proposed estimator, on average, gives less biased estimates than the ML estimator, but more biased estimates than the inversion of Kendall's τ as the GoF test fails to give close to zero p-values in case of a misspecified copula. KL distances are quite small for all estimators, though they are even smaller for the misspecified models estimated using $\hat{\theta}_{new}$ and $\hat{\theta}_{\tau}$. This exposes that, under misspecification and judging from biases, MM and as such $\hat{\theta}_{new}$ works better than ML also in this instance.

A glance at the estimated MSE efficiency (Table 21 in the Appendix) reveals that only if the sample comes from a Frank copula and Gumbel is used for estimation, applying $\hat{\theta}_{new}$

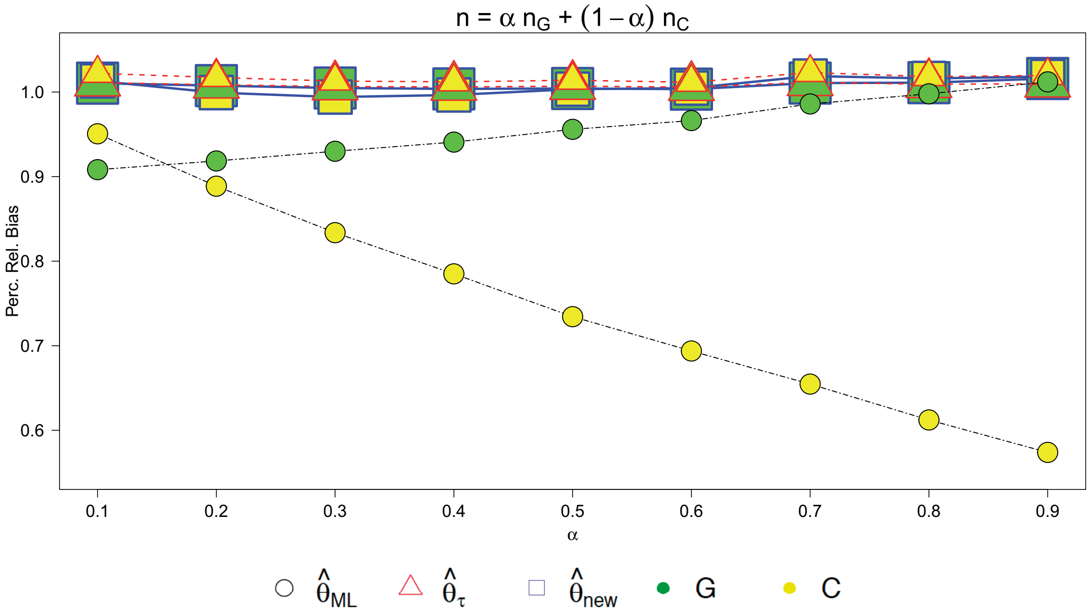

Figure 4: The figure plots the Percentage Relative Bias for each combination of estimator and estimation copula obtained from samples of convex sums of Gumbel and Clayton as function of α. The estimator is indicated by the line colour and the shape, while the estimation copula is represented by the fill colour of each shape.

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instead of ML notably reduces the MSE. For all other cases, the relative MSE is close to one.

4.2 Samples from Mixtures of Gumbel and Clayton

As Gumbel and Clayton proved to be robust against misspecification among each other for a dependence level of $\tau = 0.5$ using $\hat{\theta}_{new}$ or $\hat{\theta}_{\tau}$, 5000 samples of size $n = 100$ were drawn from a mixture of Gumbel and Clayton with $\theta = 2$ (corresponding to $\tau = 0.5$) according to Equ. (4) for $\alpha \in [0.1, 0.2, \ldots, 0.9]$. Estimation of θ was then carried out using each of the two copulas combined with each of the three estimators. Results are displayed in Tables 6 and 7 and Figure 4.

Figure 4 plots the average PRB as a function of α. Whereas the estimated PRB of $\hat{\theta}_{\tau}$ and $\hat{\theta}_{new}$ is close to one for all $\alpha's$, the estimated PRB of $\hat{\theta}_{ML}$ significantly depends on α. The less observations originate from the EC copula, the more severe is the bias. Moreover, it shows that, if Clayton is used for estimation, $\hat{\theta}_{ML}$ is very sensitive to "contamination" by observations from Gumbel. For an α of only 0.2, $\hat{\theta}_{ML}$ in combination with the Clayton copula gives more biased results than $\hat{\theta}_{ML}$ assuming a Gumbel copula. $\hat{\theta}_{ML}$ is biased up to 40 % if most observations come from Gumbel and only a small portion from Clayton. On the contrary, $\hat{\theta}_{ML}$ gives considerably less biased results when used in combination with Gumbel even if a considerable fraction of observations comes from Clayton. If 80 % of observations

α	EC	$\hat{\theta}_{ML}$			$\hat{\theta}_{\tau}$			$\hat{\theta}_{new}$			p	$\hat{\tau}$	t
		$\bar{\theta}$	KL	Bias	$\bar{\theta}$	KL	Bias	$\bar{\theta}$	KL	Bias			(in s)
0.1	Gum	1.82	7.41	-0.18	2.02	6.56	0.02	2.02	6.57	0.02	0.01	0.50	94.2
		0.17	43.16		0.22	38.49		0.22	38.55		0.04	0.05	11.0
	Clay	1.90	1.15	-0.10	2.04	1.07	0.04	2.03	1.07	0.03	0.39		37.6
		0.29	25.13		0.44	24.83		0.39	24.86		0.29		4.1
0.2	Gum	1.84	10.35	-0.16	2.02	9.25	0.02	2.02	9.25	0.02	0.03	0.50	126.1
		0.18	154.75		0.22	141.53		0.22	141.56		0.08	0.05	47.2
	Clay	1.78	11.86	-0.22	2.03	11.51	0.03	2.00	11.60	-0.00	0.29		51.3
		0.29	566.50		0.43	553.98		0.40	560.90		0.26		22.0
0.3	Gum	1.86	16.56	-0.14	2.01	15.06	0.01	2.01	15.08	0.01	0.06	0.50	83.3
		0.18	481.35		0.22	437.77		0.22	438.15		0.12	0.05	8.1
	Clay	1.67	6.54	-0.33	2.03	5.92	0.03	1.99	5.98	-0.01	0.20		35.0
		0.30	203.25		0.43	204.27		0.42	203.89		0.23		4.1
0.4	Gum	1.88	16.29	-0.12	2.01	14.41	0.01	2.01	14.74	0.01	0.10	0.50	75.9
		0.19	620.44		0.21	568.81		0.21	575.42		0.17	0.05	8.8
	Clay	1.57	267.21	-0.43	2.02	266.02	0.02	1.99	266.17	-0.01	0.12		35.7
		0.30	18518.86		0.43	18484.33		0.42	18484.67		0.18		4.8
0.5	Gum	1.91	9.87	-0.09	2.01	9.56	0.01	2.01	9.54	0.01	0.16	0.50	81.4
		0.19	356.14		0.21	356.20		0.21	356.01		0.22	0.05	9.2
	Clay	1.47	7.18	-0.53	2.03	6.60	0.03	2.01	6.62	0.01	0.07		34.1
		0.30	113.27		0.43	111.21		0.43	111.26		0.13		3.4
0.6	Gum	1.93	4.23	-0.07	2.01	4.02	0.01	2.01	4.01	0.01	0.24	0.50	73.4
		0.20	71.03		0.21	67.22		0.21	67.50		0.26	0.05	5.4
	Clay	1.39	16.20	-0.61	2.02	15.51	0.02	2.01	15.53	0.01	0.04		33.9
		0.29	454.48		0.43	451.87		0.43	451.92		0.09		2.7
0.7	Gum	1.97	27.72	-0.03	2.02	26.17	0.02	2.02	26.22	0.02	0.33	0.50	73.0
		0.20	1287.45		0.21	1201.98		0.21	1202.93		0.28	0.05	5.3
	Clay	1.31	14.90	-0.69	2.05	14.04	0.05	2.04	13.96	0.04	0.02		33.7
		0.29	379.58		0.43	368.98		0.43	368.42		0.06		2.5
0.8	Gum	2.00	1.46	-0.00	2.02	1.47	0.02	2.02	1.44	0.02	0.42	0.50	73.0
		0.21	13.75		0.21	13.92		0.21	13.80		0.29	0.05	5.0
	Clay	1.22	17.31	-0.78	2.04	16.43	0.04	2.03	16.43	0.03	0.01		33.6
		0.28	473.98		0.43	463.40		0.43	463.41		0.04		2.5
0.9	Gum	2.02	0.50	0.02	2.02	0.58	0.02	2.03	0.53	0.03	0.48	0.50	74.1
		0.21	8.40		0.22	8.62		0.22	8.49		0.29	0.05	5.5
	Clay	1.15	13.19	-0.85	2.04	12.38	0.04	2.04	12.38	0.04	0.00		33.7
		0.27	141.93		0.44	135.15		0.44	135.15		0.02		2.8

Table 6: The columns contain the sample average of obtained estimates ($\bar{\theta}$), the average of the estimated Kullback-Leibler divergence between the estimated and the true model (KL), and the average bias for the maximum likelihood estimator ($\hat{\theta}_{ML}$), the inverted Kendall's τ (θ_{τ}) estimator, and a p-value weighted average of the two ($\hat{\theta}_{new}$), respectively. 5000 data sets of size $n = 100$ were sampled from a mixture of Gumbel and Clayton both with dependence parameter $\theta = 2$ ($\tau = 0.5$) for several α. Second lines contain the corresponding standard deviations.

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come from Gumbel and Gumbel is assumed for ML, a correct estimate is obtained. In contrast, using ML and Clayton for a sample consisting of 80 % of Clayton observations, gives a biased result. This indicates, using ML, Gumbel is more robust against observations from Clayton than Clayton is against observations from Gumbel.

KL distances become really large in some cases and so do their standard deviations. Estimated KL distances for the model estimated via $\hat{\theta}_{new}$ are in all cases except for one

α	EC	Mean Squared Error			MSE efficiency	
		$\hat{\theta}_{ML}$	$\hat{\theta}_{\tau}$	$\hat{\theta}_{new}$	$\hat{\theta}_{ML}/\hat{\theta}_{new}$	$\hat{\theta}_{\tau}/\hat{\theta}_{new}$
0.1	Gum	0.06	0.05	0.05	1.29	1.00
	Clay	0.10	0.20	0.15	0.64	1.33
0.2	Gum	0.06	0.05	0.05	1.23	1.00
	Clay	0.14	0.19	0.16	0.87	1.20
0.3	Gum	0.05	0.05	0.05	1.15	1.01
	Clay	0.20	0.19	0.17	1.15	1.09
0.4	Gum	0.05	0.05	0.05	1.07	1.02
	Clay	0.27	0.18	0.18	1.54	1.03
0.5	Gum	0.05	0.05	0.05	1.02	1.02
	Clay	0.37	0.18	0.18	2.04	1.01
0.6	Gum	0.04	0.05	0.04	0.96	1.03
	Clay	0.46	0.18	0.18	2.51	1.00
0.7	Gum	0.04	0.05	0.04	0.93	1.03
	Clay	0.56	0.18	0.18	3.04	1.00
0.8	Gum	0.04	0.05	0.04	0.95	1.03
	Clay	0.68	0.18	0.18	3.67	1.00
0.9	Gum	0.05	0.05	0.05	0.95	1.02
	Clay	0.80	0.19	0.19	4.10	1.00

Table 7: The first three columns present the estimated Mean Squared Error (MSE) of the respective estimator. The last two columns contain the MSE efficiency of the ML and the Kendall's τ estimator relative to the proposed estimator for different values of α.

smaller than those for $\hat{\theta}_{ML}$ and about the same for the model estimated using $\hat{\theta}_{\tau}$, which advocates the use of $\hat{\theta}_{new}$ or $\hat{\theta}_{\tau}$ for data from the considered mixtures of Gumbel and Clayton. The GoF test rejects the null hypothesis of a Gumbel copula for $\alpha < 0.3$ and fails to reject it for Clayton. At $\alpha = 0.4$, it gives p-values of about the same size for both hypothesis. For $\alpha > 0.4$, it fails to reject the Gumbel hypothesis but rejects Clayton.

$\hat{\theta}_{new}$ returns estimates with biases of very small magnitude for all α's. Moreover, it successfully selects the less biased estimate $\hat{\theta}_{\tau}$ for almost every α and both copulas. The only exception occurs at $\alpha = 0.8$ where $\hat{\theta}_{ML}$ gives an unbiased result and $\hat{\theta}_{\tau}$ is slightly upward biased. The estimated MSE efficiency relative to $\hat{\theta}_{new}$ in Table 7 reveals that using $\hat{\theta}_{new}$ instead of $\hat{\theta}_{ML}$ is of advantage if a larger fraction of observations does not come from the copula used for estimation. In terms of relative MSE, $\hat{\theta}_{ML}$ is, in contrast, sometimes more favourable if the greater proportion of observations comes from the copula that is used for estimation. Note that the effect is not "symmetric" for Gumbel and Clayton. For Clayton, $\hat{\theta}_{ML}$ is only preferred over $\hat{\theta}_{new}$ in terms of relative MSE if $\alpha \in [0.1, 0.2]$. For Gumbel, $\hat{\theta}_{ML}$ is better in terms of relative MSE starting from an α of 0.6. Again, it shall be noted that ML gives more biased results and the relatively lower MSE of ML mainly stems from the relatively smaller standard deviations of ML compared to that of the proposed estimator $\hat{\theta}_{new}$. Comparing $\hat{\theta}_{\tau}$ and $\hat{\theta}_{new}$ in terms of MSE shows that they perform equally well yielding close to one relative MSE for almost all cases.

Summing up, $\hat{\theta}_{new}$ did not prove useful if Frank is used as the misspecified estimation copula. If Frank is the true copula though, and either Gumbel or Clayton are used for estimation, $\hat{\theta}_{new}$ equals the less biased inverted Kendall's τ estimate for all three levels of dependence.

Furthermore, $\hat{\theta}_{new}$ proved to yield unbiased results if a sample from Clayton is misspecified to originate from Gumbel or vice versa for a dependence level of $\tau = 0.5$. For the other two standard dependence levels, things are more complicated. If $\tau \in \{0.25, 0.75\}$, $\hat{\theta}_{new}$ roughly corresponds to the less biased Kendall's τ estimate for Gum-Clay and Clay-Gum respectively. This implies $\hat{\theta}_{new}$ only works the desired way for $\tau = 0.25$ if the data comes from a Gumbel copula and θ is estimated via Clayton. In contrast, for $\tau = 0.75$, $\hat{\theta}_{new}$ has only the desired effect of returning the less biased estimate if the true copula is Clayton and is misspecified for a Gumbel copula. However, $\hat{\theta}_{new}$ is still biased for these two dependence levels. As shown by the analysis of the relative MSE, using $\hat{\theta}_{new}$ instead of either $\hat{\theta}_{ML}$ or $\hat{\theta}_{\tau}$, is for most combinations either comparable or even of advantage. Only few combinations implied a worse performance in terms of MSE.

$\hat{\theta}_{new}$ also performed very well if the data comes from a mixture of Clayton and Gumbel, each with a dependence parameter $\theta = 2$. Assuming either of the two copulas and applying $\hat{\theta}_{new}$ gives a very good estimate for the true θ. Particularly, using the Gumbel and $\hat{\theta}_{new}$ outperforms $\hat{\theta}_{new}$ in combination with Clayton for almost all α.

To conclude this section, some practical issues of applying $\hat{\theta}_{new}$ shall be addressed. Conducting the GoF test is rather time-consuming due to the necessity of approximating the p-value via a parametric bootstrap procedure, especially for the Gumbel copula as the last column of Table 3 reveals. In fact, the parametric bootstrap becomes infeasible with increasing sample size and dimension as it relies on random number generation. This is, however, not a major drawback of the proposed estimator as the parametric bootstrap can be replaced by the multiplier approach. Kojadinovic and Yan (2011a) obtain an approximate p-value resorting to multiplier central limit theorems. They infer that, with an increasing sample size n, tests based on the multiplier approach are at least as powerful as tests based on the parametric bootstrap (see also Kojadinovic and Yan, 2011b).

Furthermore, it shall be remarked that computational issues might arise during the parametric bootstrap of the GoF test for the Gumbel copula when the dependence level is low and the sample size is small to moderate. The implemented optimisation method, BFGS, sometimes fails to return a result. The Nelder-Mead method proved to be more robust in such cases. However, for some random samples of size $n = 50$ the p-value could not be obtained and they had to be discarded. This, yet, does not constitute a limitation as a sample size of $n = 50$ is for the sake of a theoretical comparison and very rarely encountered in practice.

5 Empirical Example

In this section, the proposed estimator is applied to real data and its performance is compared to the standard MLE/ PMLE and the inversion of Kendall's τ. To this end, the Value-at-risk is estimated for two equally-weighted portfolios, each consisting of two assets, using the procedure described in Section 3. The first portfolio consists of Volkswagen (VW) and Thyssen-Krupp (TK) stocks, the second portfolio is composed of the Dow Jones Industrial Average (DJIA/ DJ) and the German stock index DAX. The sample period ranges from 26/08/2005 to 13/08/2015 comprising 2600 observations. Price data over this period were retrieved from Datastream and used to calculate the log-returns.

To remove the time dependence in the data, an AR-GJR-GARCH model was fit to the univariate time series. The Glosten, Jagannathan, and Runkle GARCH (Glosten et al., 1993), or short GJR-GARCH, allows the conditional variance to respond differently to past negative and positive innovations incorporating the so-called leverage effect. The volatility equation for a GJR-GARCH(1,1) model is

$$\sigma_t^2 = \omega + \alpha\epsilon_{t-1}^2 + \gamma\epsilon_{t-1}^2 \, \mathbf{I}(\epsilon_{t-1} < 0) + \beta\sigma_{t-1}^2,$$

where γ measures the effect of negative shocks compared to positive ones. Usually, it is found to be positive, implying a higher volatility increase after a negative shock compared to a positive one. To eliminate serial correlation in returns and squared returns, or more specifically, to obtain i.i.d. residuals with zero mean and unit variance, an AR(1)-GJR-GARCH(1,1) was fitted to the univariate time series of the stocks. For the same purpose, an AR(1)-GJR-GARCH(2,1) was fitted to the univariate time series of the indices. The residuals were assumed to follow a skewed generalised error distribution (SGED) (Nelson, 1991; Fernandez and Steel, 1998). The SGED not only captures the fat tails of asset returns, but also models the frequently observed skewness of financial returns (see e.g. McNeil et al., 2005).

The estimation results are given in Table 8. The chosen model successfully eliminates the autocorrelation, which can be concluded from the high p-values of the Box-Ljung tests for the series of residuals and squared residuals. The hypothesis of SGED residuals cannot be rejected by the KS test. The parameters of the marginal distributions of residuals are given in Table 9 showing that the estimated mean and variance of the residuals are approximately zero and one, respectively. The subsequent modelling uses these AR-GJR-GARCH-filtered

	$\hat{\mu}$	$\hat{\varphi}_1$	$\hat{\omega}$	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\gamma}_1$	$\hat{\gamma}_2$	$\hat{\beta}_1$	BL_{ϵ}	BL_{ϵ^2}	KS_{sged}
DAX	0.0004	-0.0234	0.0000	0.0377	0.0111	1.0000	0.6720	0.8905	0.8499	0.3212	0.5072
	0.0002	0.0178	0.0000	0.0167	0.0248	0.3117	1.3684	0.0168			
DJ	0.0002	-0.0518	0.0000	0.0306	0.0260	1.0000	1.0000	0.8696	0.9364	0.1721	0.1588
	0.0001	0.0199	0.0000	0.0137	0.0184	0.2836	0.5683	0.0144			
VW	0.0007	0.0677	0.0000	0.0991	-	0.1282	-	0.8874	0.5616	0.8544	0.7775
	0.0003	0.0182	0.0000	0.0162		0.0536		0.0174			
TK	0.0003	0.0109	0.0000	0.0617	-	0.2424	-	0.9159	0.4646	0.6858	0.7164
	0.0004	0.0200	0.0000	0.0119		0.0753		0.0139			

Table 8: Estimation results of fitting univariate AR(1)-GJR-GARCH(2,1) models to the index returns as well as of fitting univariate AR(1)-GJR-GARCH(1,1) models to the stock returns with SGED residuals. Second lines contain the corresponding standard deviations. The last three columns give the p-values of the Box-Ljung test (BL) for autocorrelation applied to the series of residuals (ϵ) and squared residuals (ϵ^2) including 12 lags and the Kolmogorov-Smirnov test (KS) for SGED.

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	$\hat{\mu}$	$\hat{\sigma}$	$\hat{\nu}$	$\hat{\xi}$
DAX	-0.0123	1.0005	1.3532	0.9042
DJ	-0.0079	0.9996	1.2807	0.9170
VW	0.0014	0.9994	1.2648	1.0187
TK	-0.0065	0.9973	1.3687	0.9806

Table 9: Parameters of the marginal distributions of residuals after fitting an AR(1)-GJR-GARCH(2,1) model to the DAX and DJ log-returns and an AR(1)-GJR-GARCH(1,1) model to the VW and TK log-returns.

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residuals.

Figure 5 shows scatterplots of the estimated residuals $\{\hat{\epsilon}_t\}_{t=1}^T$, $\hat{\epsilon}_t = (\hat{\epsilon}_{1t}, \hat{\epsilon}_{2t})^\top$ in the upper right part. The lower left part presents the estimated residuals mapped on the unit square by their empirical cdf, $\{\hat{F}(\hat{\epsilon}_t)\}_{t=1}^T$, $\hat{F}(\hat{\epsilon}_t) = (\hat{F}(\hat{\epsilon}_{1t}), \hat{F}(\hat{\epsilon}_{2t}))^\top$. The scatterplots reveal the asymmetric behaviour of the residuals, which advocates the fitting of a copula-based distribution with SGED margins to the respective two-dimensional time series of the residuals.

Again, the dependence parameter θ is estimated using the three Archimedean copulas Frank, Gumbel, and Clayton together with the three estimators, ML (IFM), the inverted Kendall's τ, and a p-value-weighted average of the two. The parameters are estimated dynamically using a moving window of size $r = 250$, $\{\hat{\epsilon}_{t=s-r+1}^{s}\}$ for $s = r, \ldots, T$. The resulting nine time series of estimated dependence parameters are illustrated in Figure 6 for each of the

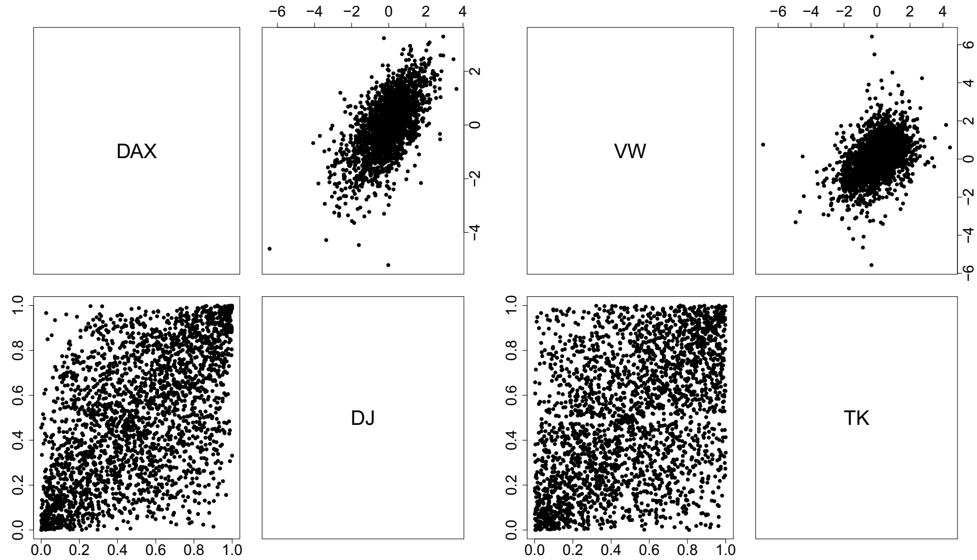

Figure 5: Scatterplots of AR-GJR-GARCH residuals (*upper triangular*) and of the AR-GJR-GARCH residuals mapped on unit square by their empirical cdf (*lower triangular*) for both portfolios.

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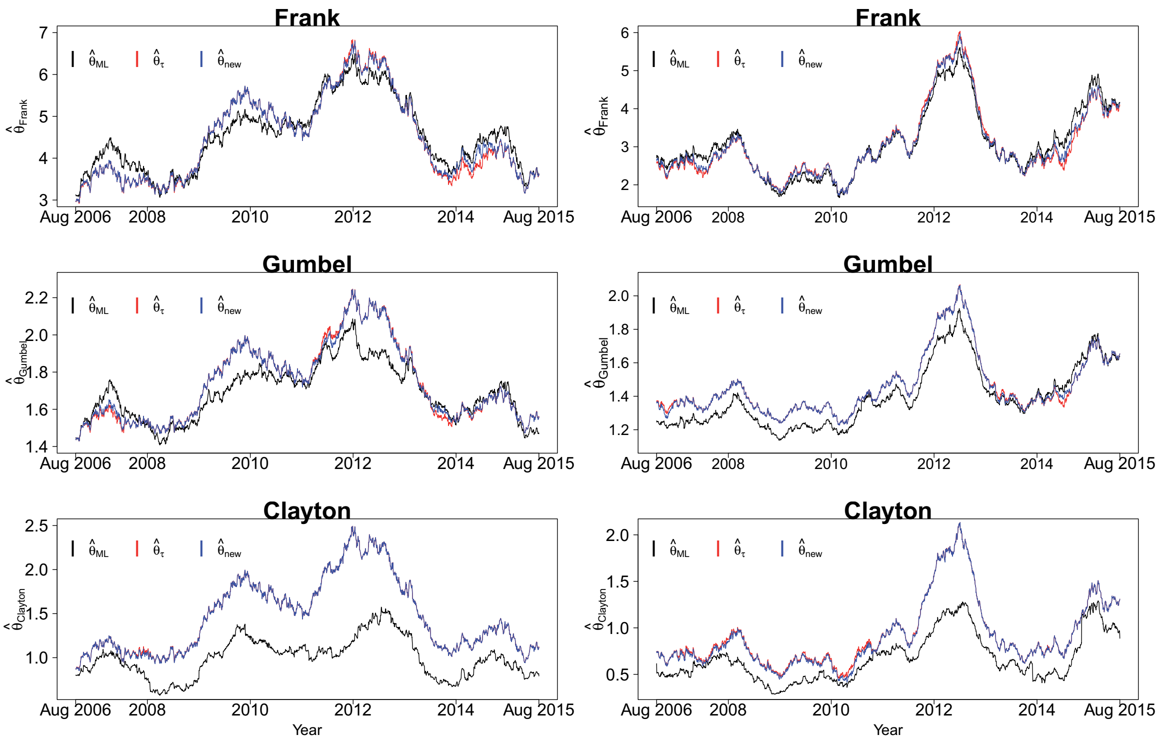

Figure 6: Dependence parameter estimates $\hat{\theta}$ for the index portfolio (*left column*) and for the stock portfolio (*right column*), estimated using ML (*black*), the inversion of Kendall's τ (*red*), and the proposed p-value weighted average of the two (*blue*) in combination with the Frank, the Gumbel, and the Clayton copula, moving window ($w = 250$).

CopDynEst

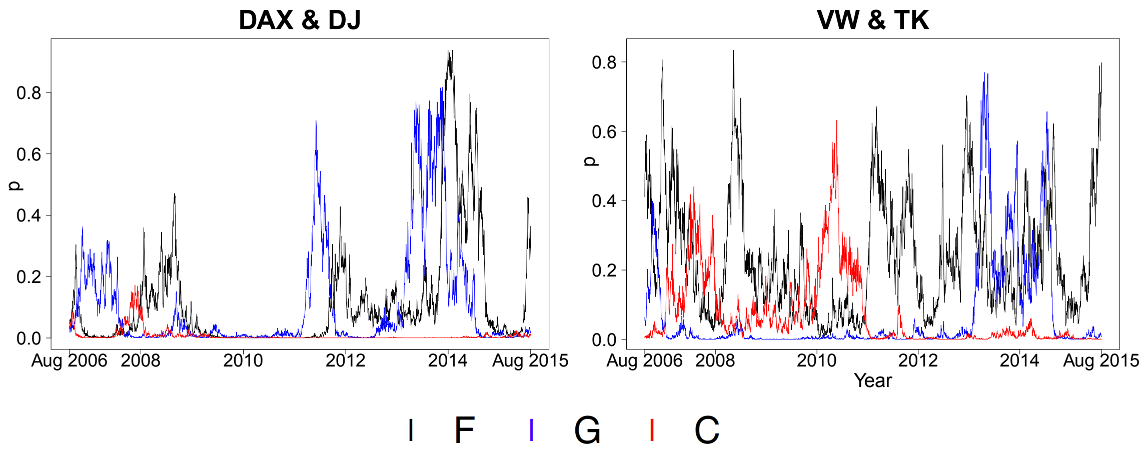

Figure 7: Time series of the p-values from the GoF test testing the Frank (*black*), the Gumbel (*blue*), and the Clayton (*red*) hypothesis.

COPp.values

two portfolios. The plots demonstrate the dramatic change in the dependence parameter over time justifying the dynamic approach. Furthermore, they illustrate that $\hat{\theta}_{new}$ and $\hat{\theta}_{\tau}$ mostly coincide. The further analysis of the difference between the estimators will be deferred until later.

Figure 7 plots the time series of the p-values from the GoF test testing the Frank, the Gumbel, and the Clayton hypothesis. For the index portfolio, the dependence between the univariate residual time series is first best described by Gumbel, followed by a short period of Clayton and then by Frank. For a considerable time period in the middle of the sampling period and at the end, none of the three copulas is adequate to model the dependence between DAX and DJ. Towards the end, p-values rise indicating it is more likely the respective copula is the data-generating one and Gumbel and Frank cannot be rejected as true copula. Throughout most of the sample period, the GoF test clearly selects one copula as the best fit returning very low p-values for the other two. Only towards the end, it gives rather high p-values for Frank and Gumbel. Clayton is hardly ever not rejected. For the stock portfolio, p-values are higher for all three copulas and the Clayton hypothesis is less often rejected compared to the index portfolio. The GoF test gives a less clear answer of which copula fits best returning rather high p-values for more than one copula. From the beginning of 2011 to the end of 2013 as well as towards the end of the sampling period, however, the GoF test only yields high p-values for Frank. The size of the p-values is reflected in the evolution of $\hat{\theta}_{new}$ over time.

Figure 8 plots the difference between the estimates obtained by the ML (IFM) estimator and the proposed estimator, $\hat{\theta}_{ML} - \hat{\theta}_{new}$, as well as the difference between the estimates

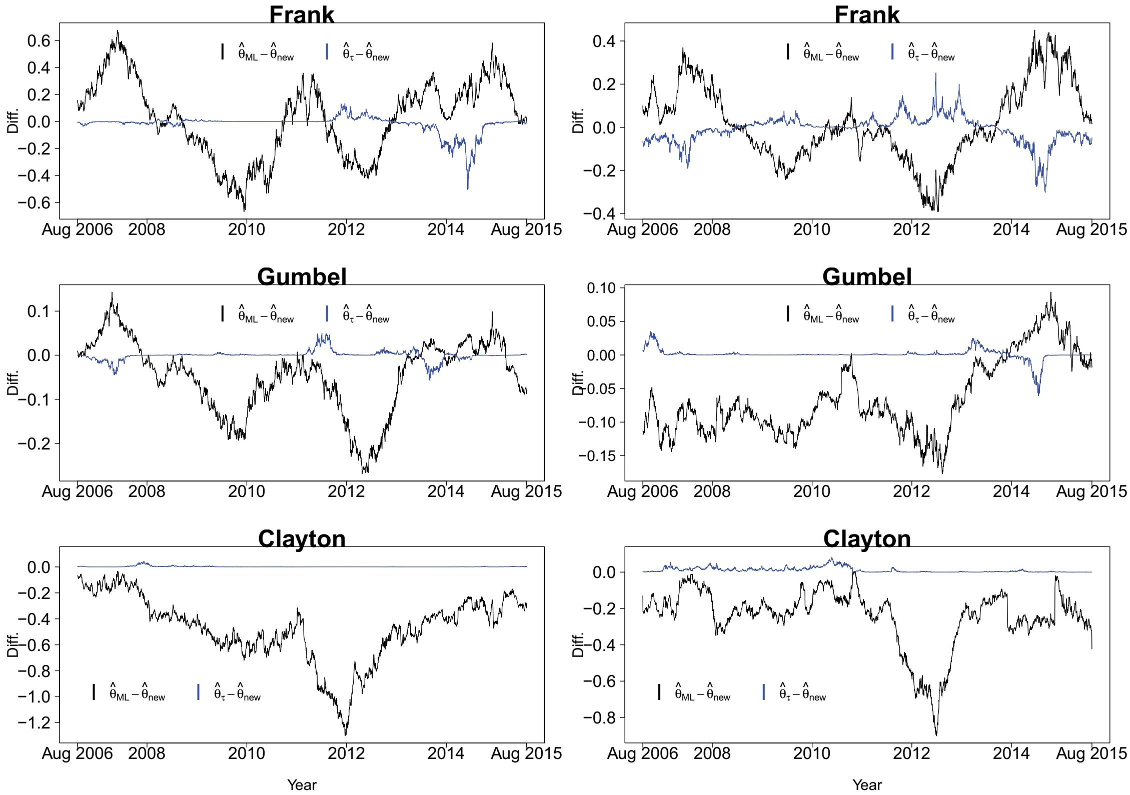

Figure 8: Plots of $\hat{\theta}_{ML} - \hat{\theta}_{new}$ and of $\hat{\theta}_{\tau} - \hat{\theta}_{new}$ over time for each of the three copulas and both portfolios, index portfolio (*left column*) and stock portfolio (*right column*).

COPthetasDiff

	Frank		Gumbel		Clayton	
	$\hat{\theta}_{ML} - \hat{\theta}_{new}$	$\hat{\theta}_{\tau} - \hat{\theta}_{new}$	$\hat{\theta}_{ML} - \hat{\theta}_{new}$	$\hat{\theta}_{\tau} - \hat{\theta}_{new}$	$\hat{\theta}_{ML} - \hat{\theta}_{new}$	$\hat{\theta}_{\tau} - \hat{\theta}_{new}$
DAX & DJ	0.0247	-0.0184	-0.0481	-0.0004	-0.4756	0.0020
	0.2848	0.0657	0.0822	0.0111	0.2528	0.0047
TK & VW	0.0166	-0.0098	-0.0651	0.0009	-0.2449	0.0106
	0.1755	0.0622	0.0563	0.0082	0.1603	0.0136

Table 10: Summary statistics of the differences between the two standard estimators and the proposed estimator. Second lines contain the corresponding standard deviations.

COPthetasDiff

using Kendall's τ and the proposed estimator, $\hat{\theta}_{\tau} - \hat{\theta}_{new}$. Table 10 give mean and standard deviations of the differences. The plots show that for both portfolios the proposed estimator closely tracks $\hat{\theta}_{\tau}$. For the Clayton family $\hat{\theta}_{\tau}$ and $\hat{\theta}_{new}$ are identical for the index portfolio throughout the whole period. Deviations from $\hat{\theta}_{\tau}$ are also negligible for the stock portfolio. Also for the Gumbel family, $\hat{\theta}_{\tau}$ and $\hat{\theta}_{new}$ do not differ much for both portfolios except for short periods of small deviations. In contrast, greater and more frequent deviations of $\hat{\theta}_{new}$ from $\hat{\theta}_{\tau}$ can be observed for the Frank copula, especially for the stock portfolio. The difference between $\hat{\theta}_{ML}$ and $\hat{\theta}_{new}$ is for both portfolios and all copula families greater than $\hat{\theta}_{\tau} - \hat{\theta}_{new}$ throughout the whole period. These observations are also reflected in the higher standard deviations of $\hat{\theta}_{ML} - \hat{\theta}_{new}$ compared to $\hat{\theta}_{\tau} - \hat{\theta}_{new}$ presented in Table 10.

The results of the backtesting for the index portfolio are summarised in Table 11. Table 12 displays the backtesting results for the stock portfolio. The left side of the respective table shows the empirical level $\hat{\alpha}$ estimated by the exceedances ratio in Equ. (31). The smaller the difference between the empirical level and the theoretical level, the better is the performance of the model and/ or the estimator. The right side of the tables presents the relative difference e between the theoretical α and the empirical $\hat{\alpha}$ given according to Equ. (32). For both portfolios, $\hat{\alpha}$ is closest to the theoretical value α for the Clayton family. Exhibiting the lowest relative differences e for all levels of α and all estimators in both portfolios, the Clayton family clearly outperforms the Gumbel and the Frank family. This is a consequence of the fact that Clayton is the only copula with lower tail dependence allowing for joint losses, which Gumbel and Frank do not. So even if Clayton does not fit the data very well, as indicated by the GoF test, it better describes the tails.

Comparing the performance of the three estimators does not allow such a straightforward conclusion. The bold numbers in the tables mark the lowest e for each level of α within the respective copula family. For both portfolios and for an α of 0.1 %, the estimation method does not influence the performance of the model in case of the Clayton family. For that level using the Clayton copula, all estimators perform equally. Leaving this combination aside, 11 "wins" in terms of e can be assigned for each table. Table 13 summarises the wins across all copula families and levels of α. For the index portfolio, $\hat{\theta}_{\tau}$ and $\hat{\theta}_{new}$ outperform $\hat{\theta}_{ML}$ achieving the same e at the 1 % level using the Clayton copula. In that case both estimators were given a win. However, giving neither of them a win would not really change the picture as can be deduced from the numbers in parenthesis in Table 13.

In these two applications, $\hat{\theta}_{\tau}$ and $\hat{\theta}_{new}$ outperform $\hat{\theta}_{ML}$ on average. Overall, $\hat{\theta}_{\tau}$ turned out to perform best being slightly ahead of $\hat{\theta}_{new}$. Note, however, that the proposed estimator, performs very well for the index data in combination with the Frank copula clearly outperforming $\hat{\theta}_{ML}$ and $\hat{\theta}_{\tau}$ for all levels of α. Nonetheless, even though $\hat{\alpha}$ obtained by employing $\hat{\theta}_{new}$ relatively deviates the least from the true α, the Frank copula still significantly fails to hold the nominal level for $\alpha = 0.5\%$ and $\alpha = 0.1\%$.

		5 %	1 %	0.5 %	0.1 %	5 %	1 %	0.5 %	0.1 %
Frank	$\hat{\theta}_{ML}$	0.0617	0.0209	0.0136	0.0043	0.2340	1.0851	1.7234	3.2553
	$\hat{\theta}_{\tau}$	0.0604	0.0209	0.0128	0.0051	0.2085	1.0851	1.5532	4.1064
	$\hat{\theta}_{new}$	0.0583	0.0174	0.0115	0.0034	**0.1660**	**0.7447**	**1.2979**	**2.4043**
Gumbel	$\hat{\theta}_{ML}$	0.0634	0.0162	0.0085	0.0021	0.2681	**0.6170**	**0.702**	**1.1277**
	$\hat{\theta}_{\tau}$	0.0626	0.0174	0.0089	0.0026	**0.2511**	0.7447	0.7872	1.5532
	$\hat{\theta}_{new}$	0.0643	0.0170	0.0106	0.0030	0.2851	0.7021	1.1277	1.9787
Clayton	$\hat{\theta}_{ML}$	0.0553	0.0111	0.0038	0.0009	0.1064	0.1064	**0.2340**	0.1489
	$\hat{\theta}_{\tau}$	0.0485	0.0094	0.0034	0.0009	**0.0298**	**0.0638**	0.3191	0.1489
	$\hat{\theta}_{new}$	0.0528	0.0094	0.0034	0.0009	0.0553	**0.0638**	0.3191	0.1489

Table 11: Backtesting results for the **index portfolio** for each copula in combination with each of the three estimators: exceedances ratio $\hat{\alpha}$ (*left*) and relative distance e between α and $\hat{\alpha}$ (*right*).

CopVaRBackTesRes1

		5 %	1 %	0.5 %	0.1 %	5 %	1 %	0.5 %	0.1 %
Frank	$\hat{\theta}_{ML}$	0.0596	0.0170	0.0098	0.0047	0.1915	0.7021	**0.9574**	3.6809
	$\hat{\theta}_{\tau}$	0.0591	0.0149	0.0123	0.0047	0.1830	**0.4894**	1.4681	3.6809
	$\hat{\theta}_{new}$	0.0583	0.0157	0.0102	0.0043	**0.1660**	0.5745	1.0426	**3.2553**
Gumbel	$\hat{\theta}_{ML}$	0.0600	0.0179	0.0106	0.0038	0.2000	0.7872	1.1277	2.8298
	$\hat{\theta}_{\tau}$	0.0574	0.0166	0.0094	0.0034	**0.1489**	**0.6596**	**0.8723**	**2.4043**
	$\hat{\theta}_{new}$	0.0609	0.0187	0.0102	0.0043	0.2170	0.8723	1.0426	3.2553
Clayton	$\hat{\theta}_{ML}$	0.0532	0.0106	0.0060	0.0021	0.0638	**0.0638**	0.1915	1.1277
	$\hat{\theta}_{\tau}$	0.0502	0.0111	0.0081	0.0021	**0.0043**	0.1064	0.6170	1.1277
	$\hat{\theta}_{new}$	0.0528	0.0119	0.0051	0.0021	0.0553	0.1915	**0.0213**	1.1277

Table 12: Backtesting results for the **stock portfolio** for each copula in combination with each of the three estimators: exceedances ratio $\hat{\alpha}$ (*left*) and relative distance e between α and $\hat{\alpha}$ (*right*).

CopVaRBackTesRes2

	DAX & DJ	VW & TK	Total	%
$\hat{\theta}_{ML}$	4 (4)	2	6 (6)	26.1 (28,6)
$\hat{\theta}_{\tau}$	3 (2)	6	9 (8)	39.1 (38.1)
$\hat{\theta}_{new}$	5 (4)	3	8 (7)	34,8 (33.3)

Table 13: Wins of each estimator across all copula families and levels of α.

6 Conclusion

This thesis studied the effect of misspecification among the three Archimedean copula families Frank, Gumbel, and Clayton on the dependence parameter estimation for two dimensions. In addition to the two standard approaches for copula calibration, the maximum likelihood estimator and the method of moments estimator based on Kendall's τ, a p-value weighted average of the two was proposed and studied. As maximum likelihood is well known to heavily rely on the correctness of the assumed model to have the preferable properties of consistency and asymptotic efficiency, whereas, the inversion of Kendall's τ is not asymptotically efficient, however, proved to be less sensitive to copula misspecification in some cases, the purpose of the proposed estimator is to check the validity of the assumed model and to place more weight on that estimate that is more likely to have preferable properties. To assess the performance of the proposed estimator and to compare it to the two standard estimators, a comprehensive simulation study was conducted considering two sample sizes as well as samples from a mixture of Clayton and Gumbel for $\tau = 0.5$.

As expected, the ML estimates indeed are biased in case of copula misspecification. For Clayton and Gumbel, the inversion of Kendall's τ returns less biased results compared to ML under misspecification except for two cases. For $\tau = 0.5$, the inverted Kendall's τ even gives unbiased results under mutual copula misspecification. The proposed estimator also proved to yield unbiased results under copula misspecification if the sample originates either from a Gumbel or a Clayton copula with dependence parameter $\theta = 2$, corresponding to a Kendall's τ of 0.5, and the respective other copula is erroneously employed for estimation.

Moreover, for a sample stemming from Frank and being misspecified as having a Gumbel or Clayton dependence structure, the proposed estimator returns the less biased inverted Kendall's τ estimate for all three studied levels of dependence. Furthermore, simply using the Gumbel or Clayton copula for estimating the dependence parameter from a sample of the considered Gumbel-Clayton mixtures, the suggested estimator clearly outperformed ML giving unbiased results.

In the given application of estimating the Value-at-Risk of two bivariate portfolios using the three copulas in combination with each of the three estimators, the proposed estimator again outperformed the ML estimator on the whole, though the Kendall's τ estimator performed slightly better.

The contribution of the proposed estimator is to lay more weight on ML when the needed conditions for ML to be consistent and efficient are very likely to hold, and conversely, it attaches more weight to the, in some cases, more robust MM estimator if the necessary assumptions for ML are less likely to hold. Of course, a careful analysis preceding the estimation could select the appropriate estimator. However, the proposed p-value weighted average of the two standard estimators wraps this up in one step and is therefore more convenient. This is especially the case for a dynamic estimation, like in the estimation of the Value-at-risk, where a careful selection of the estimator in every step would have been unfeasible. Moreover, in a case where the GoF test only gives a moderate p-value and makes a clear decision whether the conditions for ML are fulfilled or not hard, the proposed estimator is a compromise, which proved to be of advantage in case of misspecification and small samples.

Still it shall be clearly remarked that the estimator may not be applied blindly and at any case. Some diagnostic checking needs to be done in advance. For a dependence level around $\tau = 0.5$ and data presumably either from Clayton or Gumbel or a mixture of the two, the proposed estimator is a good choice. Furthermore, the reader shall be reminded that model misspecification might in some cases not impact the estimation of the dependence parameter, as shown in this particular study, yet, employing a misspecified model in forecasting or other further modelling might be very misleading as important features of the assumed model and the true model might differ.

The approach of using a p-value weighted average of an ML estimate and an MM estimate in the estimation of copulas could only be examined to a small extent in this thesis leaving a lot of space for future research. The concluding paragraphs are therefore dedicated to pointing out some future research questions. For the future application of the proposed estimator, it is vital to identify the interval around $\tau = 0.5$ for which the proposed estimator, or ultimately the inverted Kendall's τ estimator, give good estimates under mutual misspecification of Clayton and Gumbel. To really be able to judge the performance and the reliability as well as the appropriateness of the estimator in applications, the proposed estimator needs to be tried further with real world data.

For the performance of the proposed estimator, the performance of the GoF test is crucial. Yet there exists no single GoF test that dominates the others under all circumstances. A way to make use of several tests at a time are so-called hybrid tests (see e.g. Zhang et al., 2013). Different GoF tests and their influence on the performance of the proposed estimator need to be further investigated.

As Kendall's τ is a measure of dependence between two random variables, the simple

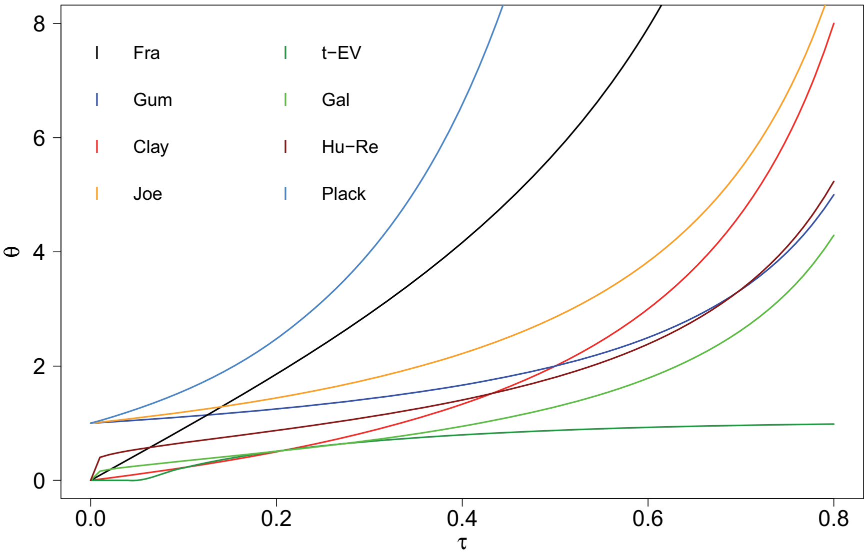

Figure 9: $\tau(\theta)$-Functions for the Frank (Fra), the Gumbel (Gum), the Clayton (Clay), the Joe (Joe), the t-EV (t-EV), the Galambos (Gal), the Husler-Reiss (Hu-Re) and the Plackett copula for τ over the interval $[0, 0.8]$.

COPtautotheta

inversion of Kendall's τ is limited to bivariate copula models. Some procedures have been developed to make use of this dependence measure in the estimation of multivariate elliptical copula models (Demarta and McNeil, 2005; Genest et al., 2007). The investigation of the proposed estimator could as such be extended to the multivariate case.

Future research on this topic should additionally broaden the investigated copulas to other classes and families as well as study further dependence levels. Figure 9 illustrates that many intersection points of the $\tau(\theta)$-functions for different copulas exist. For example, the $\tau(\theta)$-functions of the Gumbel and the Husler-Reiss copula as well as of the Clayton and the Husler-Reiss copula share common points. Future research could investigate the effect of misspecification among these copulas on the estimation of the dependence parameter.

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A Supplementary Figures - Simulation Study

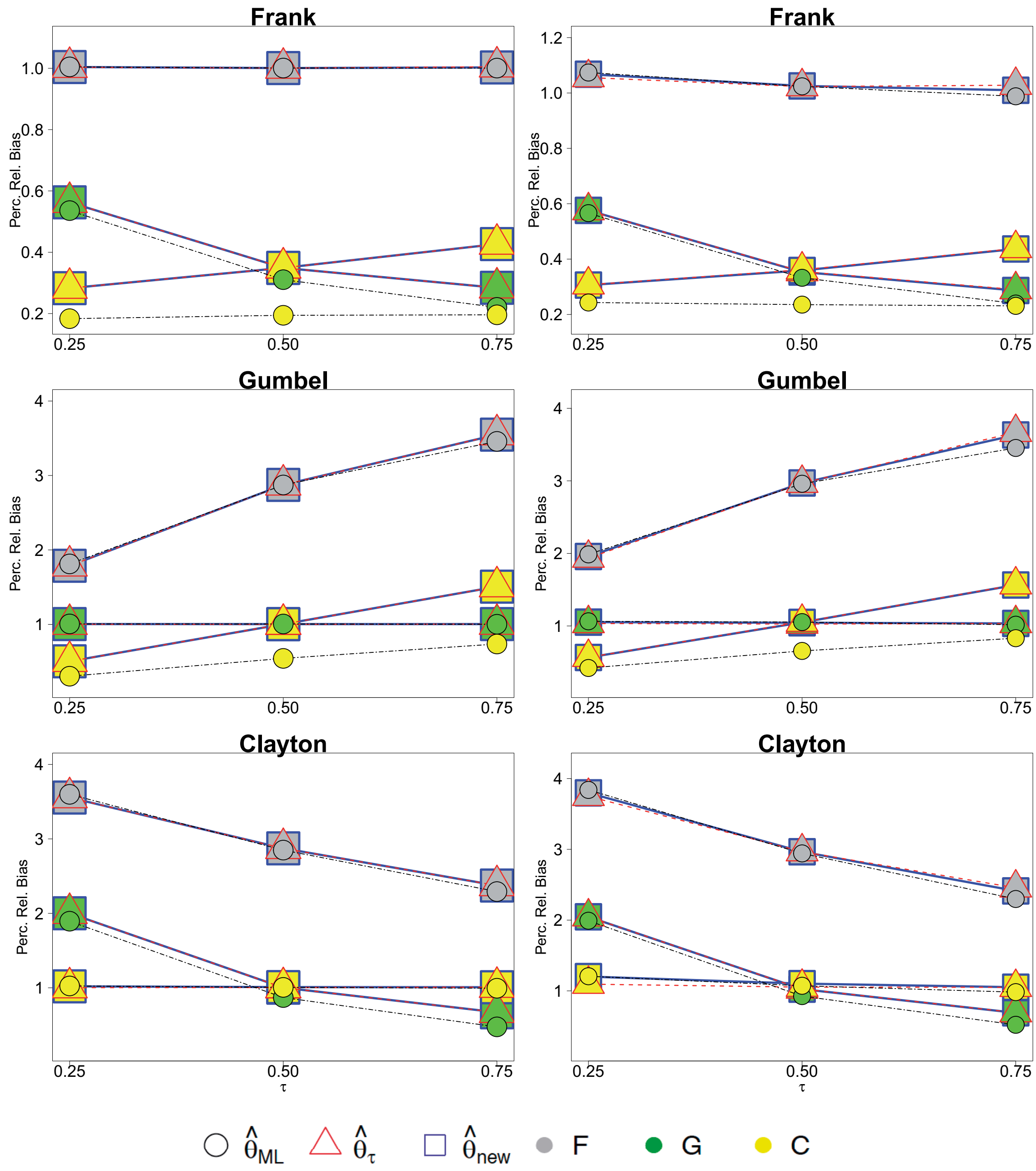

Figure 10: The Figure plots the estimated Percentage Relative Bias for each combination of estimator and estimation copula as function of the dependence level τ for a sample size of $n = 500$ (*left column*) and a sample size of $n = 50$ (*right column*). The estimator is indicated by the line colour and the shape, while the estimation copula is represented by the fill colour of each shape. The titles of the plots state the true copula. F, G and C stand for Frank, Gumbel and Clayton, respectively.

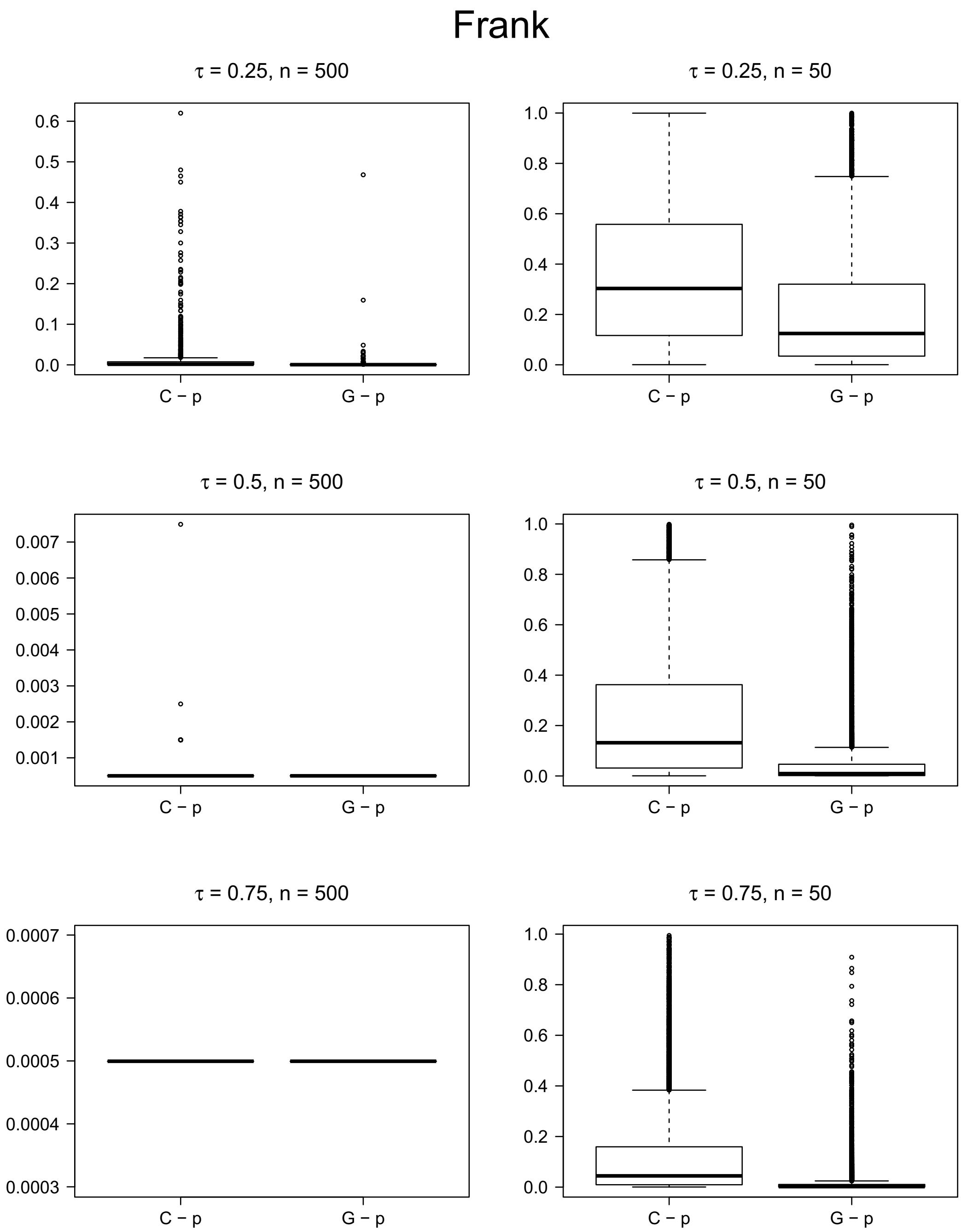

Figure 11: Distribution of the p-values testing for Clayton and Gumbel when Frank is the true copula.

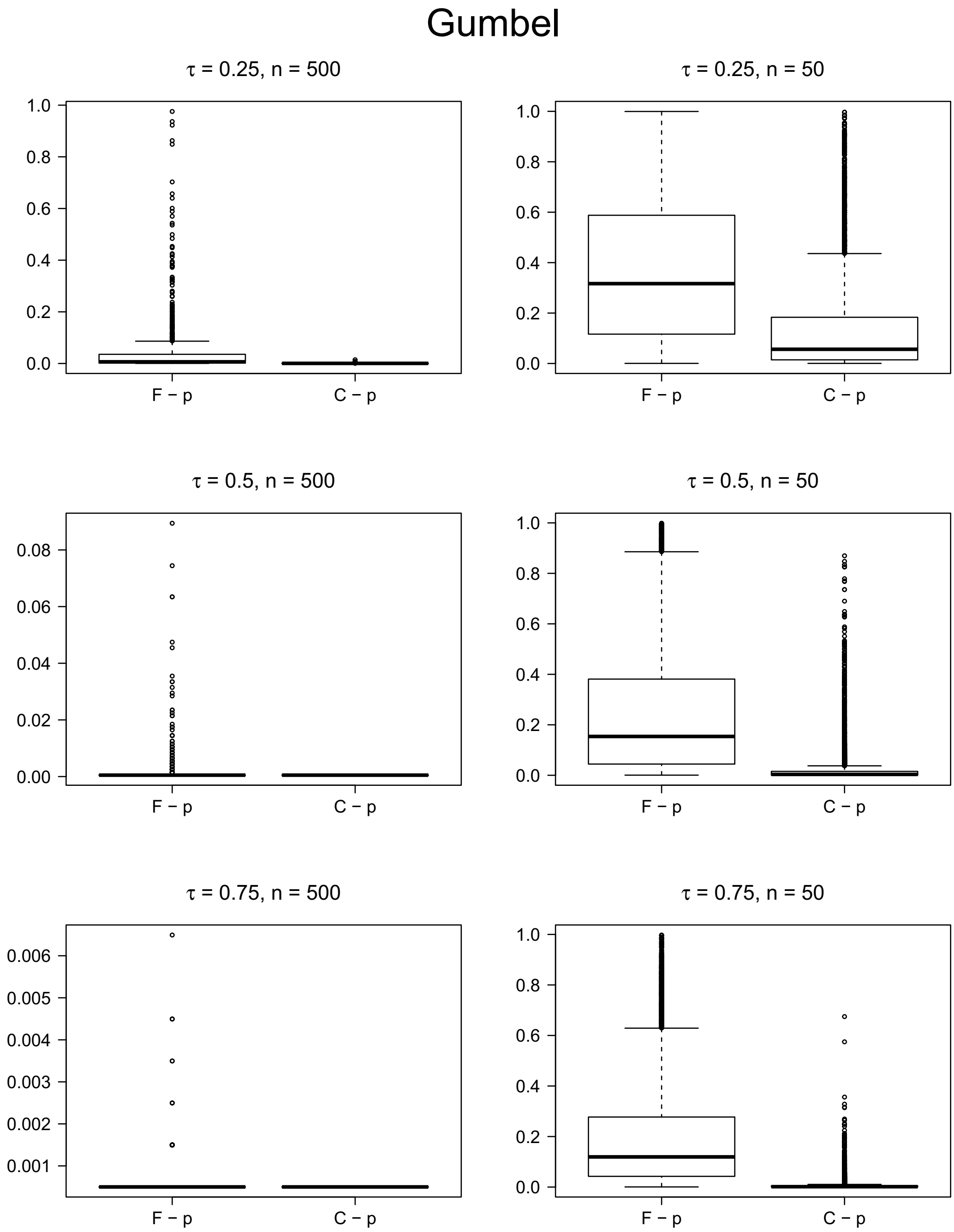

Figure 12: Distribution of the p-values testing for Frank and Clayton when Gumbel is the true copula.

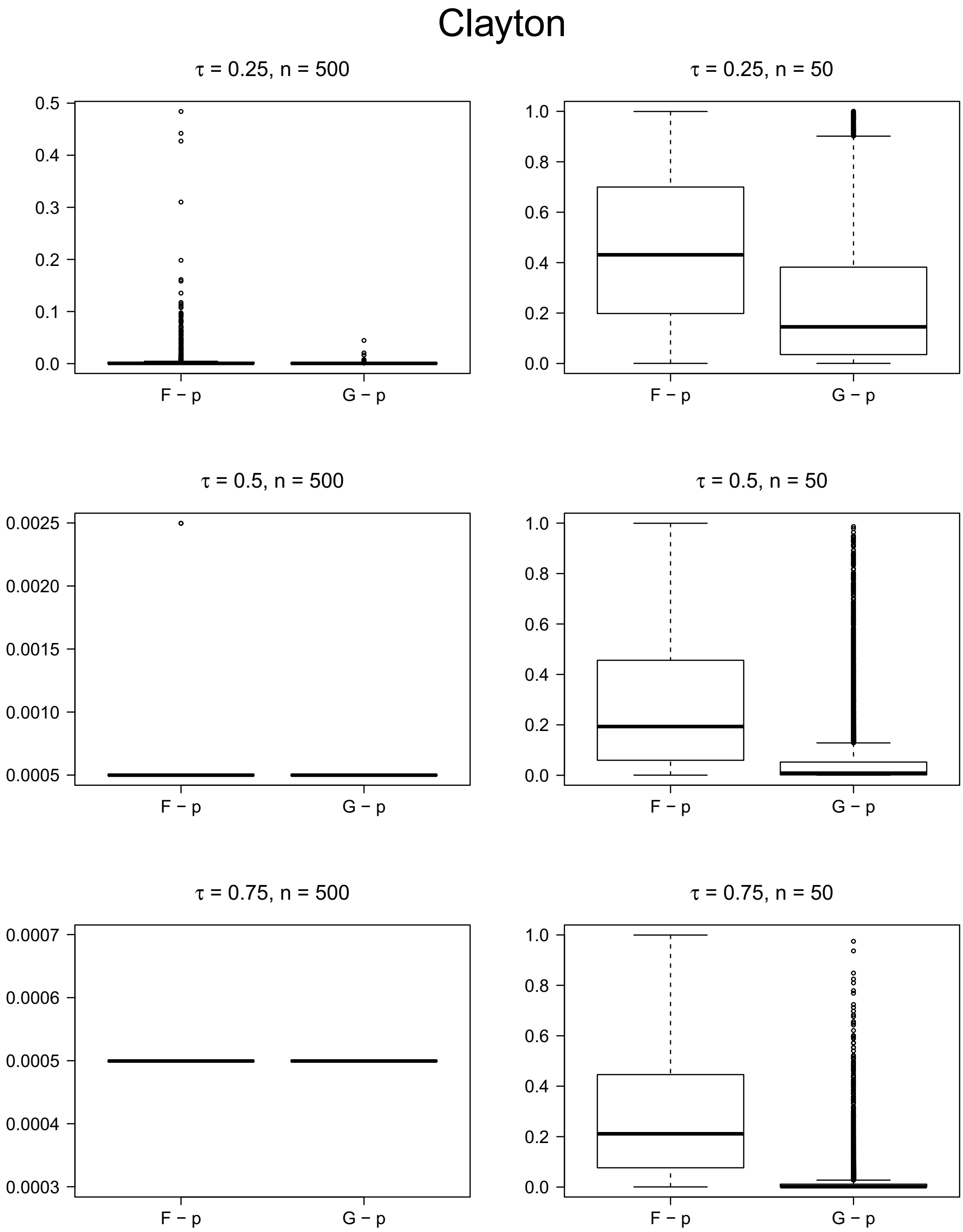

Figure 13: Distribution of the p-values testing for Frank and Gumbel when Clayton is the true copula.

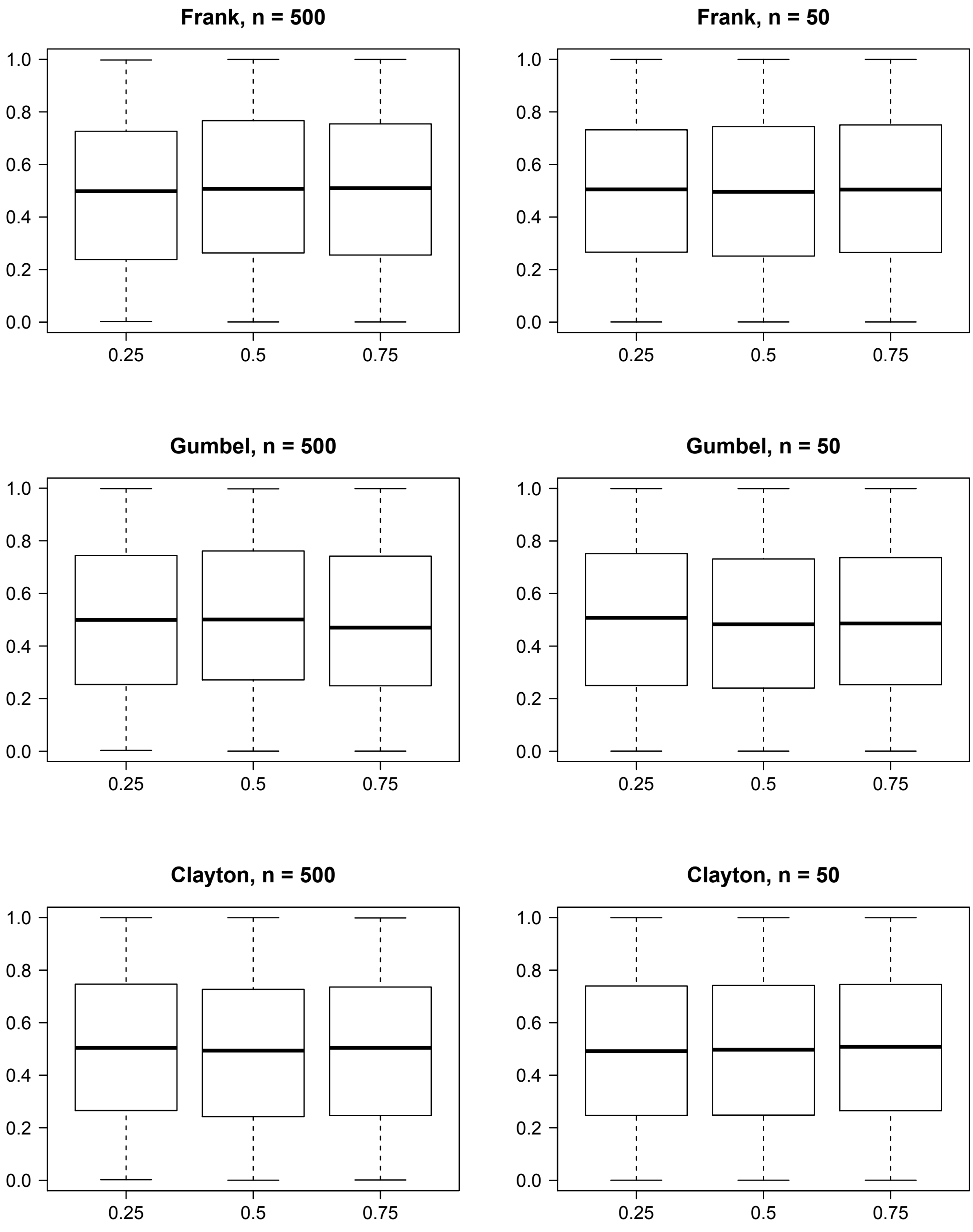

Figure 14: Distribution of the p-values testing for the true copula.

B Supplementary Tables - Simulation Study

B.1 Large Samples

TC	EC	$\tau = 0.25$			$\tau = 0.5$			$\tau = 0.75$		
		$\hat{\theta}_{ML}$	$\hat{\theta}_{\tau}$	$\hat{\theta}_{new}$	$\hat{\theta}_{ML}$	$\hat{\theta}_{\tau}$	$\hat{\theta}_{new}$	$\hat{\theta}_{ML}$	$\hat{\theta}_{\tau}$	$\hat{\theta}_{new}$
Fra	Fra	1.07	1.06	1.07	1.02	1.02	1.03	0.99	1.03	1.01
		0.38	0.38	0.38	0.22	0.22	0.22	0.16	0.18	0.17
	Gum	0.57	0.58	0.58	0.33	0.36	0.35	0.24	0.29	0.29
		0.06	0.07	0.07	0.04	0.05	0.05	0.03	0.04	0.04
	Clay	0.24	0.31	0.31	0.24	0.36	0.36	0.23	0.44	0.44
		0.10	0.13	0.13	0.06	0.10	0.10	0.06	0.09	0.09
Gum	Fra	1.99	1.94	1.96	2.96	2.97	2.97	3.45	3.67	3.63
		0.73	0.73	0.72	0.69	0.73	0.72	0.70	0.80	0.77
	Gum	1.06	1.03	1.05	1.05	1.03	1.04	1.02	1.03	1.03
		0.13	0.13	0.13	0.16	0.16	0.16	0.17	0.20	0.19
	Clay	0.42	0.57	0.57	0.66	1.05	1.05	0.83	1.57	1.56
		0.19	0.26	0.25	0.20	0.32	0.32	0.23	0.39	0.39
Clay	Fra	3.84	3.75	3.80	2.94	2.97	2.97	2.30	2.46	2.41
		1.45	1.46	1.46	0.70	0.75	0.74	0.49	0.55	0.52
	Gum	1.99	2.05	2.05	0.93	1.03	1.02	0.52	0.69	0.69
		0.23	0.26	0.25	0.13	0.17	0.16	0.09	0.14	0.13
	Clay	1.21	1.09	1.20	1.07	1.05	1.10	0.98	1.05	1.05
		0.47	0.51	0.49	0.29	0.33	0.31	0.23	0.27	0.25

Table 14: The table presents the estimated percentage relative bias for each of the three estimators and each dependence level. Second lines contain the corresponding standard deviations.

TC	EC	$\tau = 0.25$			$\tau = 0.5$			$\tau = 0.75$		
		$\hat{\theta}_{ML}$	$\hat{\theta}_{\tau}$	$\hat{\theta}_{new}$	$\hat{\theta}_{ML}$	$\hat{\theta}_{\tau}$	$\hat{\theta}_{new}$	$\hat{\theta}_{ML}$	$\hat{\theta}_{\tau}$	$\hat{\theta}_{new}$
Fra	Fra	0.08	0.08	0.08	0.13	0.13	0.13	0.43	0.45	0.43
	Gum	1.21	1.08	1.08	15.64	13.95	13.96	121.57	102.47	102.48
	Clay	3.76	2.90	2.90	21.36	13.96	13.97	129.28	65.79	65.81
Gum	Fra	1.27	1.19	1.20	14.18	14.21	14.21	97.38	104.72	104.71
	Gum	0.00	0.00	0.00	0.01	0.01	0.01	0.04	0.05	0.04
	Clay	0.87	0.45	0.45	0.85	0.03	0.03	1.17	4.30	4.30
Clay	Fra	3.09	2.99	2.99	13.86	14.22	14.22	60.89	68.72	68.72
	Gum	0.36	0.45	0.45	0.07	0.01	0.01	9.94	3.95	3.95
	Clay	0.01	0.01	0.01	0.03	0.04	0.03	0.17	0.20	0.18

Table 15: The table presents the estimated MSE for each of the three estimators for each dependence level.

B.2 Small Samples

TC	EC	$\tau = 0.25$			$\tau = 0.5$			$\tau = 0.75$		
		$\hat{\theta}_{ML}$	$\hat{\theta}_{\tau}$	$\hat{\theta}_{new}$	$\hat{\theta}_{ML}$	$\hat{\theta}_{\tau}$	$\hat{\theta}_{new}$	$\hat{\theta}_{ML}$	$\hat{\theta}_{\tau}$	$\hat{\theta}_{new}$
Fra	Fra	1.07	1.06	1.07	1.02	1.02	1.03	0.99	1.03	1.01
		0.38	0.38	0.38	0.22	0.22	0.22	0.16	0.18	0.17
	Gum	0.57	0.58	0.58	0.33	0.36	0.35	0.24	0.29	0.29
		0.06	0.07	0.07	0.04	0.05	0.05	0.03	0.04	0.04
	Clay	0.24	0.31	0.31	0.24	0.36	0.36	0.23	0.44	0.44
		0.10	0.13	0.13	0.06	0.10	0.10	0.06	0.09	0.09
Gum	Fra	1.99	1.94	1.96	2.96	2.97	2.97	3.45	3.67	3.63
		0.73	0.73	0.72	0.69	0.73	0.72	0.70	0.80	0.77
	Gum	1.06	1.03	1.05	1.05	1.03	1.04	1.02	1.03	1.03
		0.13	0.13	0.13	0.16	0.16	0.16	0.17	0.20	0.19
	Clay	0.42	0.57	0.57	0.66	1.05	1.05	0.83	1.57	1.56
		0.19	0.26	0.25	0.20	0.32	0.32	0.23	0.39	0.39
Clay	Fra	3.84	3.75	3.80	2.94	2.97	2.97	2.30	2.46	2.41
		1.45	1.46	1.46	0.70	0.75	0.74	0.49	0.55	0.52
	Gum	1.99	2.05	2.05	0.93	1.03	1.02	0.52	0.69	0.69
		0.23	0.26	0.25	0.13	0.17	0.16	0.09	0.14	0.13
	Clay	1.21	1.09	1.20	1.07	1.05	1.10	0.98	1.05	1.05
		0.47	0.51	0.49	0.29	0.33	0.31	0.23	0.27	0.25

Table 16: The table presents the estimated percentage relative bias for each of the three estimators and each dependence level. Second lines contain the corresponding standard deviations.

TC	EC	$\tau = 0.25$			$\tau = 0.5$			$\tau = 0.75$		
		$\hat{\theta}_{ML}$	$\hat{\theta}_{\tau}$	$\hat{\theta}_{new}$	$\hat{\theta}_{ML}$	$\hat{\theta}_{\tau}$	$\hat{\theta}_{new}$	$\hat{\theta}_{ML}$	$\hat{\theta}_{\tau}$	$\hat{\theta}_{new}$
Fra	Fra	0.85	0.84	0.84	1.56	1.68	1.63	5.20	6.65	5.72
	Gum	1.08	1.04	1.04	14.72	13.77	13.84	115.82	101.17	102.11
	Clay	3.28	2.80	2.80	19.36	13.75	13.85	118.90	64.60	65.02
Gum	Fra	2.67	2.49	2.56	17.28	17.63	17.62	104.02	124.14	120.35
	Gum	0.04	0.03	0.03	0.11	0.11	0.11	0.49	0.63	0.57
	Clay	0.66	0.45	0.45	0.63	0.43	0.43	1.29	7.61	7.55
Clay	Fra	4.52	4.31	4.43	17.08	17.72	17.62	69.19	87.39	81.41
	Gum	0.46	0.52	0.52	0.09	0.11	0.11	8.58	4.09	4.12
	Clay	0.12	0.12	0.13	0.36	0.45	0.44	1.95	2.76	2.38

Table 17: The table presents the estimated MSE for each of the three estimators for each dependence level.

TC	EC	$\tau = 0.25$		$\tau = 0.5$		$\tau = 0.75$	
		$\hat{\theta}_{ML}/\hat{\theta}_{new}$	$\hat{\theta}_{\tau}/\hat{\theta}_{new}$	$\hat{\theta}_{ML}/\hat{\theta}_{new}$	$\hat{\theta}_{\tau}/\hat{\theta}_{new}$	$\hat{\theta}_{ML}/\hat{\theta}_{new}$	$\hat{\theta}_{\tau}/\hat{\theta}_{new}$
Fra	Fra	1.01	0.99	0.96	1.03	0.91	1.16
	Gum	1.04	1.00	1.06	1.00	1.13	0.99
	Clay	1.17	1.00	1.40	0.99	1.83	0.99
Gum	Fra	1.04	0.97	0.98	1.00	0.86	1.03
	Gum	1.10	0.90	1.00	0.97	0.86	1.11
	Clay	1.47	1.00	1.46	1.00	0.17	1.01
Clay	Fra	1.02	0.97	0.97	1.01	0.85	1.07
	Gum	0.89	1.00	0.77	1.02	2.08	0.99
	Clay	0.92	0.95	0.83	1.03	0.82	1.16

Table 18: The table presents the estimated MSE efficiency of the ML and the Kendall's τ estimator relative to the proposed estimator for all dependence levels.

TC	EC	$\hat{\theta}_{ML}$			$\hat{\theta}_{\tau}$			$\hat{\theta}_{new}$			p	$\hat{\tau}$	t
		$\bar{\theta}$	Bias	MSE	$\bar{\theta}$	Bias	MSE	$\bar{\theta}$	Bias	MSE			(in s)
Fra	Fra	2.55	-0.02	0.18	2.50	-0.01	0.13	2.53	-0.01	0.16	0.50	0.26	26.5
($\theta = 2.37$)		0.91	0.05		0.90	0.05		0.90	0.05		0.28	0.08	2.80
	Gum	1.34	0.02	-1.03	1.36	0.02	-1.01	1.37	0.02	-1.01	0.36		63.5
		0.15	0.06		0.16	0.06		0.16	0.06		0.28		9.3
	Clay	0.58	0.05	-1.80	0.73	0.05	-1.64	0.73	0.05	-1.65	0.22		22.9
		0.24	0.06		0.31	0.07		0.31	0.07		0.23		1.6
Gum	Fra	2.65	2.09	1.32	2.58	2.10	1.25	2.61	2.10	1.28	0.37	0.26	15.5
($\theta = 1.33$)		0.97	26.70		0.97	26.71		0.97	26.71		0.29	0.09	0.7
	Gum	1.42	-0.12	0.09	1.38	-0.04	0.05	1.40	-0.09	0.07	0.50		35.9
		0.18	0.88		0.17	0.72		0.17	0.81		0.29		3.6
	Clay	0.56	2.46	-0.77	0.76	2.39	-0.58	0.75	2.39	-0.58	0.14		13.7
		0.25	28.91		0.34	28.37		0.34	28.38		0.19		0.6
Clay	Fra	2.56	3.11	1.89	2.50	3.12	1.83	2.53	3.12	1.87	0.46	0.26	20.0
($\theta = 0.67$)		0.97	63.46		0.98	63.48		0.97	63.49		0.29	0.09	1.4
	Gum	1.33	2.45	0.66	1.36	2.35	0.70	1.36	2.35	0.70	0.24		48.2
		0.16	50.34		0.17	48.57		0.17	48.73		0.25		7.0
	Clay	0.80	-0.17	0.14	0.73	0.04	0.06	0.80	-0.10	0.13	0.49		17.1
		0.31	3.05		0.34	6.72		0.33	4.42		0.29		1.2
Fra	Fra	5.87	-0.01	0.14	5.87	-0.01	0.13	5.88	-0.01	0.15	0.50	0.50	17.3
($\theta = 5.74$)		1.24	0.10		1.29	0.10		1.27	0.10		0.28	0.07	1.7
	Gum	1.91	0.13	-3.83	2.04	0.12	-3.70	2.03	0.12	-3.71	0.23		34.6
		0.24	0.16		0.29	0.17		0.28	0.17		0.25		3.3
	Clay	1.35	0.32	-4.38	2.07	0.37	-3.66	2.06	0.37	-3.68	0.05		14.0
		0.37	0.20		0.57	0.35		0.57	0.34		0.11		1.3
Gum	Fra	5.92	9.32	3.92	5.94	9.31	3.94	5.94	9.31	3.94	0.25	0.50	30.6
($\theta = 2.0$)		1.38	172.57		1.46	172.80		1.45	172.75		0.25	0.08	18.1
	Gum	2.11	-0.17	0.11	2.05	-0.07	0.05	2.09	-0.14	0.09	0.49		57.1
		0.32	4.68		0.32	4.55		0.32	4.52		0.29		31.6
	Clay	1.31	11.31	-0.69	2.11	10.79	0.11	2.10	10.79	0.10	0.02		22.8
		0.40	189.88		0.65	185.33		0.65	185.36		0.06		12.9
Clay	Fra	5.89	28.44	3.89	5.93	28.37	3.93	5.93	28.38	3.93	0.29	0.50	21.9
($\theta = 2.0$)		1.41	614.33		1.49	612.00		1.47	612.13		0.27	0.08	1.0
	Gum	1.85	16.80	-0.15	2.05	14.78	0.05	2.05	14.81	0.05	0.06		44.1
		0.25	320.09		0.33	274.10		0.33	274.34		0.13		2.3
	Clay	2.14	-0.20	0.14	2.11	0.10	0.11	2.20	-0.16	0.20	0.50		16.6
		0.59	16.66		0.66	13.10		0.63	15.85		0.29		0.8
Fra	Fra	13.97	0.02	-0.17	14.54	-0.02	0.40	14.28	-0.00	0.14	0.51	0.75	34.7
($\theta = 14.14$)		2.27	0.20		2.55	0.20		2.39	0.20		0.28	0.04	13.4
	Gum	3.39	0.70	-10.75	4.10	0.78	-10.04	4.05	0.77	-10.09	0.13		64.0
		0.48	0.62		0.63	0.86		0.62	0.85		0.19		23.7
	Clay	3.27	1.51	-10.87	6.20	2.28	-7.94	6.17	2.27	-7.97	0.02		25.6
		0.83	0.85		1.25	1.89		1.25	1.89		0.05		10.5
Gum	Fra	13.81	83.50	9.81	14.68	82.70	10.68	14.53	82.76	10.53	0.19	0.75	29.4
($\theta = 4.0$)		2.78	4778.08		3.19	4740.50		3.08	4741.27		0.20	0.05	3.0
	Gum	4.07	0.01	0.07	4.14	0.04	0.14	4.14	-0.03	0.14	0.50		56.1
		0.70	13.37		0.78	15.02		0.74	14.56		0.28		3.7
	Clay	3.33	96.31	-0.67	6.27	91.59	2.27	6.26	91.59	2.26	0.01		22.8
		0.91	5293.24		1.57	5026.80		1.56	5026.93		0.02		1.8
Clay	Fra	13.79	103.12	7.79	14.74	101.96	8.74	14.47	102.19	8.47	0.29	0.75	28.8
($\theta = 6.0$)		2.92	3877.78		3.32	3839.04		3.10	3843.01		0.25	0.05	1.5
	Gum	3.12	49.73	-2.88	4.15	39.19	-1.85	4.14	39.23	-1.86	0.02		58.5
		0.55	1558.36		0.82	1128.89		0.81	1129.10		0.06		3.0
	Clay	5.89	0.09	-0.11	6.30	0.02	0.30	6.31	-0.11	0.31	0.51		22.5
		1.39	15.34		1.64	20.41		1.51	18.77		0.28		1.4

Table 19: The columns contain the sample average of obtained estimates ($\bar{\theta}$), the average of the estimated Kullback-Leibler divergence between the estimated and the true model (KL), and the average bias for the maximum likelihood estimator $\hat{\theta}_{ML}$, the inverted Kendall's τ ($\hat{\theta}_{\tau}$), and for the p-value weighted average of the two ($\hat{\theta}_{new}$), respectively. 10 000 data sets of size $n = 50$ were sampled from the Frank (Fra), the Gumbel (Gum), and the Clayton (Clay) copula for $\tau = 0.25$ (*upper part*), $\tau = 0.5$ (*middle part*), and $\tau = 0.75$ (*lower part*). Second lines contain the corresponding standard deviations.

B.3 Frank - Gumbel, $\tau = 0.1255$

TC	EC	Percentage Relative Bias		
		$\hat{\theta}_{ML}$	$\hat{\theta}_{\tau}$	$\hat{\theta}_{new}$
Frank	Frank	1.00	1.00	1.00
		0.24	0.24	0.24
	Gumbel	0.97	1.00	1.00
		0.03	0.03	0.03
Gumbel	Frank	1.02	1.01	1.01
		0.24	0.24	0.24
	Gumbel	1.01	1.00	1.00
		0.03	0.03	0.03

Table 20: The columns contain the estimated percentage relative bias for the three estimators for Frank and Gumbel and the dependence level $\tau = 0.1255$. Second lines contain the corresponding standard deviations.

TC	EC	Mean Squared Error			MSE efficiency	
		$\hat{\theta}_{ML}$	$\hat{\theta}_{\tau}$	$\hat{\theta}_{new}$	$\hat{\theta}_{ML}/\hat{\theta}_{new}$	$\hat{\theta}_{\tau}/\hat{\theta}_{new}$
Frank	Frank	0.07	0.07	0.07	1.00	1.00
	Gumbel	0.00	0.00	0.00	1.50	0.96
Gumbel	Frank	0.08	0.08	0.08	1.02	1.00
	Gumbel	0.00	0.00	0.00	0.97	1.04

Table 21: The first three columns present the estimated Mean Squared Error (MSE) of the respective estimator and the last two contain the MSE efficiency of the ML and the Kendall's τ estimator relative to the proposed estimator for Frank and Gumbel and the dependence level $\tau = 0.1255$.

B.4 Mixtures of Gumbel and Clayton

α	EC	Percentage Relative Bias		
		$\hat{\theta}_{ML}$	$\hat{\theta}_{\tau}$	$\hat{\theta}_{new}$
0.1	Gum	0.91	1.01	1.01
		0.09	0.11	0.11
	Clay	0.95	1.02	1.01
		0.15	0.22	0.19
0.2	Gum	0.92	1.01	1.01
		0.09	0.11	0.11
	Clay	0.89	1.02	1.00
		0.15	0.22	0.20
0.3	Gum	0.93	1.01	1.00
		0.09	0.11	0.11
	Clay	0.83	1.01	0.99
		0.15	0.22	0.21
0.4	Gum	0.94	1.01	1.00
		0.09	0.11	0.11
	Clay	0.78	1.01	1.00
		0.15	0.21	0.21
0.5	Gum	0.96	1.01	1.00
		0.10	0.11	0.11
	Clay	0.73	1.01	1.00
		0.15	0.21	0.21
0.6	Gum	0.97	1.01	1.00
		0.10	0.11	0.11
	Clay	0.69	1.01	1.00
		0.15	0.21	0.21
0.7	Gum	0.99	1.01	1.01
		0.10	0.11	0.10
	Clay	0.65	1.02	1.02
		0.14	0.21	0.21
0.8	Gum	1.00	1.01	1.01
		0.10	0.11	0.11
	Clay	0.61	1.02	1.02
		0.14	0.21	0.21
0.9	Gum	1.01	1.01	1.02
		0.11	0.11	0.11
	Clay	0.57	1.02	1.02
		0.13	0.22	0.22

Table 22: The table presents the estimated percentage relative bias for each of the three estimators for data coming from convex sums of Gumbel and Clayton. Second lines contain the corresponding standard deviations.

Declaration of Authorship

I hereby confirm that I have authored this Master's thesis independently and without use of others than the indicated sources. All passages which are literally or in general matter taken out of publications or other sources are marked as such.

Berlin, December 09, 2015

Verena Weber